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EUCLIDE's ELEMENTS;

The whole Fifteen BOOKS Compendiously Demonstrated.

WITH

ARCHIMEDES *Theorems* Of the Sphere and Cylinder, investigated by the *Method of Indivisibles.*

By ISAAC BARROW, *D. D. Late Master of* Trinity *College in* Cambridge.

To which is added in this Edition,

EUCLIDE's DATA with *Marinus's* Preface.

And a Brief TREATISE of

REGULAR SOLIDS.

Καθαρμοὶ ψυχῆς λογικῆς εἰσὶν αἱ μαθηματικαὶ ἐπιστήμαι.

LONDON: Printed and Sold by *W. Redmayne* in *Fewen-street*, *R. Mount* on *Tower-hill*, and *J.* and *B. Sprint* in *Little-britain.* 1714.

BRITISH MUSEUM

To the READER.

IF you are desirous, Courteous Reader, to know what I have perform'd in this Edition of the Elements of Euclide, I shall here explain it to you in short, according to the nature of the Work. I have endeavor'd to attain two ends chiefly; the first, to be very perspicuous, and at the same time so very brief, that the Book may not swell to such a Bulk, as may be troublesome to carry about one, in which I think I have succeeded, unless in my absence the Printer's care should frustrate my Design. Some of a brighter Genius, and endued with greater Skill, may have demonstrated most of these Propositions with more nicety, but perhaps none with more succinctness than I have; especially since I alter'd nothing in the number and order of the Author's Propositions; nor presum'd either to take the liberty of rejecting, as less necessary, any of them, or of reducing some of the easier sort into the rank of Axioms, as several have done; and among others, that most expert Geometrician A. Tacquetus C. (whom I the more willingly name, because I think it is but civil to acknowledge that I have imitated him in some Points) after whose most acurate Edition I had no Thoughts of attempting any thing of this nature, till I consider'd that this most learned Man thought fit to publish only eight of Euclide's Books, which he took the pains to explain and embellish, having in a manner rejected and undervalued the other seven, as less appertaining to the Elements of Geometry. But my Province was

originally quite different, not that of writing the Elements of Geometry, after what method ſoever I pleas'd, but of demonſtrating, in as few words as poſſible I cou'd, the whole Works of Euclide. *As to four of the Books,* viz. *the ſeventh, eighth, ninth, and tenth, altho' they don't ſo nearly appertain to the Elements of plain and ſolid Geometry, as the ſix precedent and the two ſubſequent, yet none of the more skilful Geometricians can be ſo ignorant as not to know that they are very uſeful for Geometrical matters, not only by reaſon of the mighty near affinity that is between Arithmetick and Geometry, but alſo for the knowledge of both meaſurable and unmeaſurable Magnitudes, ſo exceeding neceſſary for the Doctrine of both plain and ſolid figures. Now the noble Contemplation of the five Regular Bodies that is contain'd in the three laſt Books, cannot without great Injuſtice be pretermitted, ſince that for the ſake thereof our* στοιχειωτὴς, *being a Philoſopher of the Platonic Sect, is ſaid to have compos'd this univerſal Syſtem of Elements; as* Proclus lib. 2. *witneſſeth in theſe Words,* Ὅθεν δὴ καὶ τῆς συμπάσης στοιχειώσεως τέλος προσῆκε τὴν τῶν καλουμένων πλατωνικῶν χημάτων σύστασιν. *Beſides, I eaſily perſwaded my ſelf to think, that it would not be unacceptable to any Lover of theſe Sciences to have in his Poſſeſſion the whole Euclidean Work, as it is commonly cited and celebrated by all Men. Wherefore I reſolv'd to omit no Book or Propoſition of thoſe that are found in* P. Herigonius's *Edition, whoſe Steps I was oblig'd cloſely to follow, by reaſon I took a Reſolution to make uſe of moſt of the Schemes of the ſaid Book, very well foreſeeing that time would not allow me to form*

new

new ones, tho' ſometimes I choſe rather to do it. For the ſame reaſon I was willing to uſe for the moſt part Euclide's *own Demonſtrations, having only expreſs'd them in a more ſuccinct Form, unleſs perhaps in the ſecond, thirteenth, and very few in the ſeventh, eighth, and ninth Book, in which it ſeem'd not worth my while to deviate in any particular from him. Therefore I am not without good hopes that as to this part I have in ſome meaſure ſatisfied both my own Intentions, and the Deſire of the Studious. As for ſome certain Problems and Theorems that are added in the Scholions (or ſhort Expoſitions) either appertaining (by reaſon of their frequent Uſe) to the nature of theſe Elements, or conducing to the ready Demonſtration of thoſe things that follow, or which do intimate the reaſons of ſome principal Rules of practical Geometry, reducing them to their original Fountains, theſe I ſay, will not, I hope, make the Book ſwell to a Size beyond the deſign'd Proportion.*

The other Butt, which I levell'd at, is to content the Deſires of thoſe who are delighted more with ſymbolical than verbal Demonſtrations. In which kind, whereas moſt among us are accuſtom'd to the Symbols of Gulielmus Oughtredus, *I therefore thought beſt to make uſe, for the moſt part, of his. None hitherto (as I know of) has attempted to interpret and publiſh* Euclide *after this manner, except* P. Herigonius; *whoſe Method (tho' indeed moſt excellent in many things, and very well accommodated for the particular purpoſe of that moſt ingenious Man) yet ſeems in my Opinion to labour under a double Defect. Firſt, in regard that, altho' of two or more Propoſitions, produ-*

ced

ced for the Proof of any one Problem or Theorem, the former don't always depend of the latter, yet it don't readily enough appear either from the order of each, or by any other manner, when they agree together, and when not; wherefore for want of the Conjunctions and Adjectives, ergo, rursus, *&c. many difficulties and occasions of doubt do often arise in reading, especially to those that are Novices. Besides it frequently happens, that the said Method cannot avoid superfluous Repetitions, by which the Demonstrations are oftentimes render'd tedious, and sometimes also more intricate; which Faults my Method doth easily remedy by the arbitrary mixture of both Words and Signs. Therefore let what has been said, touching the Intention and Method of this little Work, suffice. As to the rest, whoever covets to please himself with what may be said, either in Praise of the Mathematicks in general, or of Geometry in particular, or touching the History of these Sciences, and consequently of* Euclide *himself, (who digested those Elements) and others* ἐξωτερικὰ *of that kind, may consult other Interpreters. Neither will I (as if I were afraid least these my Endeavors may fall short of being satisfactory to all Persons) alledge as an Excuse (tho' I may very lawfully do it) the want of due time which ought to be employ'd in this Work, nor the Interruption occasion'd by other Affairs, nor yet the want of requisite help for these Studies nor several other things of the like nature. But what I have here employ'd my Labour and Study in for the Use of the ingenuous Reader, I wholly submit to his Censure and Judgment, to approve if useful, or reject if otherwise.*

J. B.

Ad amiciſſimum Virum, *I. C.* de EUCLIDE contracto, Εὐσυναγωγός.

FActum bene! didicit Laconice loqui
Senex profundus, & aphoriſmos induit.
Immenſa dudum margo commentarii
Diagramma circuit minutum; utque Inſulæ
Problema breve natabat in vaſto mari.
Sed unda jam detumuit; & gloſſa arctior
Stringit Theoremata: minoris anguli
Lateribus ecce totus Euclides *jacet,*
Incluſus olim velut Homerus in nuce;
Pluteoque ſarcina modo qui incubuit, levis
En ſit manipulus. Pelle in exigua latet
Ingens Matheſis, matris utero Hercules,
In glande quercus, vel Ithaca Eurus in pila.
Nec mole dum decreſcit, uſu ſit minor;
Quin auctior jam evadit, & cumulatius
Contracta prodeſt erudita pagina.
Sic ubere magis liquor è preſſo affluit;
Sic pleniori vaſa inundat ſanguinis
Torrente cordis Syſtole; ſic fuſius
Procurrit æquor ex Abyla anguſtiis.
Tantilli operis ars tanta referenda unice eſt
BAROVIANO *nomini, ac ſolertiæ.*
Sublimis euge mentis ingenium potens!
Cui invitum nil, arduum eſſe nil ſolet;
Sic uſque pergas proſpero conamine,
Radiuſque multum debeat ac abacus tibi;
Sic creſcat indies feracior ſeges,
Simili colonum germine aſſiduo beans.
Specimen futuræ meſſis hic fiet labor.
Magnæque famæ illuſtria hæc præludia.
Juvenis dedit qui tanta, quid dabit ſenex?

Car. Robotham, CANTAB.
Coll. Trin. Sen. Soc.

The

The Explication of the Signs or Characters.

=	Signifies	Equal.
⊏		Greater.
⊐		Leſſer.
+		More, or to be added.
−		Leſs, or to be ſubtracted.
∴		The Differences, or Exceſs; Alſo, that all the quantities which follow, are to be ſubtracted, the Signs not being changed.
×		Multiplication, or the Drawing one ſide of a Rectangle into another. The ſame is denoted by the Conjunction of letters; as AB = A × B.
√		The Side or Root of a Square, or Cube, &c.
Q & q		A Square.
C & c		A Cube.
Q. Q.		The ratio of a ſquare number to a ſquare number.

Other Abbreviations of words, where-ever they accur, the Reader will without trouble underſtand of himſelf; ſaving ſome few, which, being of leſs general uſe, we refer to be explained in their places.

The

THE FIRST BOOK OF EUCLIDE'S ELEMENTS.

Definitions.

I. A Point is that which has no part.

II. A Line is a longitude without latitude.

III. The ends, or limits, of a line are Points.

IV. A Right Line is that which lies equally betwixt its Points.

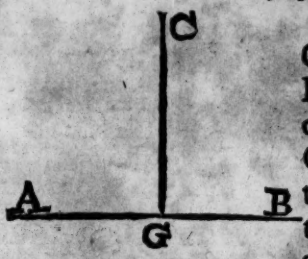
V. A Superficies is that which has only longitude and latitude.

VI. The extremes, or limits, of a Superficies are lines.

VII. A plain Superficies is that which lies equally betwixt its lines.

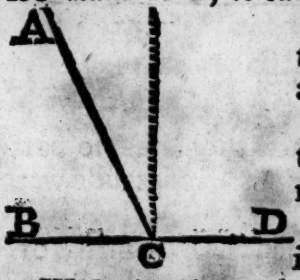
VIII. A plain Angle is the inclination of two lines the one to the other, the one touching the other in the ſame plain, yet not lying in the ſame ſtrait line.

IX. And if the lines which contain the Angle be right lines, it is called a right-lined Angle.



X. When a right line CG ſtanding upon a right line AB, makes the angles on either ſide thereof, CGA, CGB, equal one to the other, then both thoſe equal angles are right angles; and the right line CG, which ſtandeth on the other, is termed a Perpendicular to that (AB) whereon it ſtandeth.

Note, *When ſeveral angles meet at the ſame point (as at G) each particular angle is deſcribed by three letters; whereof the middle letter ſheweth the angular point, and the two other letters the lines that make that angle: As the angle which the right lines CG, AG make at G, is called CGA, or AGC.*



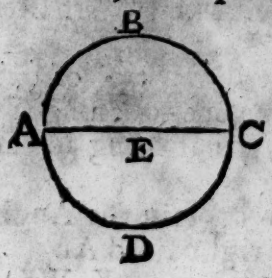
XI. An obtuſe angle is that which is greater than a right angle; as ACD.

XII. An acute angle is that which is leſs than a right angle; as ACB.

XIII. A Limit, or Term, is the end of any thing.

XIV. A Figure is that which is contained under one or more terms.

XV. A Circle is a plain figure contained under one line, which is called a Circumference; unto which all lines drawn from one point within the figure, and falling upon the circumference thereof, are equal the one to the other.



XVI. And that point is called the Center of the Circle.

XVII. A Diameter of a circle is a right line drawn through the center thereof, and ending at the circumference on either ſide, divi-

dividing the circle into two equal parts.

XVIII. A Semicircle is a figure which is contained under the diameter, and under that part of the circumference which is cut off by the diameter.

In the circle EABCD, E is the center, AC the diameter, ABC the ſemicircle.

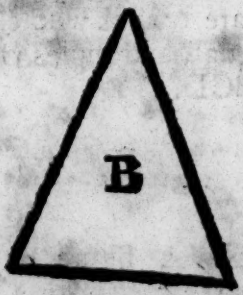
XIX. Right-lined figures are ſuch as are contained under right lines.

XX. Three-ſided or Trilateral figures are ſuch as are contained under three right lines.

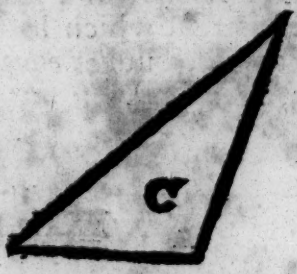
XXI. Four-ſided or Quadrilateral figures are ſuch as are contained under four right lines.

XXII. Many-ſided figures are ſuch as are contained under more right lines than four.

XXIII. Of Trilateral figures, that is an Equilateral Triangle, which hath three equal ſides; as the Triangle A.



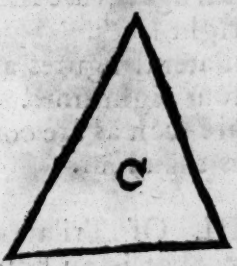
XXIV. Iſoſceles is a Triangle which hath only two ſides equal; as the Triangle B.



XXV. Scalenum is a Triangle whoſe three ſides are all unequal; as C.

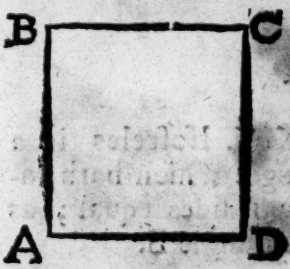
XXVI. Of these Trilateral figures, a right-angled Triangle is that which has one right angle; as the Triangle A.

XXVII. An Amblygonium, or obtuse-angled Triangle, is that which has one angle obtuse; as B.

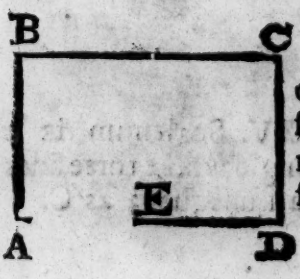


XXVIII. An Oxygonium, or acute-angled Triangle, is that which has three acute angles; as C.

An Equiangular, or equal-angled figure is that whereof all the angles are equal. Two figures are equiangular, if the several angles of the one figure be equal to the several angles of the other. The same is to be understood of Equilateral figures.

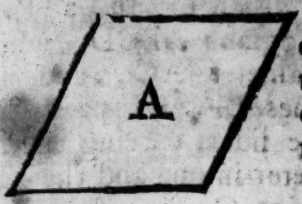


XXIX. Of Quadrilateral, or four-sided figures, a Square is that whose sides are equal, and angles right; as ABCD.

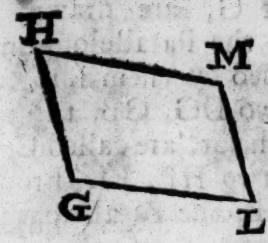


XXX. A figure on the one part longer, or a long square, is that which hath right angles, but not equal sides; as ABCD.

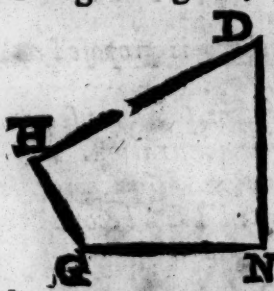
XXXI. A



XXXI. A Rhombus, or diamond-figure, is that which has four equal sides, but is not right-angled; as A.

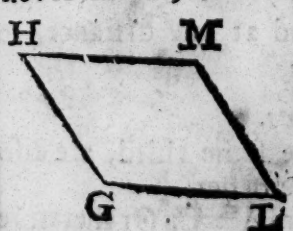


XXXII. A Rhomboides, or diamond-like figure, is that whose opposite sides, and opposite angles, are equal; but has neither equal nor right angles; as GLMH.

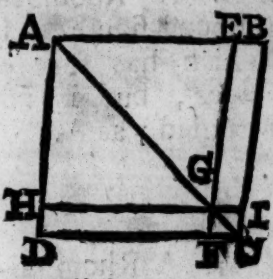


XXXIII. All other quadrilateral figures besides these are called *Trapezia*, or Tables; as GNDH.

XXXIV. Parallel, or equidistant right lines are such, which being in the same superficies, if infinitely produced, would never meet; as A and B.



XXXV. A Parallelogram is a quadrilateral figure, whose opposite sides are parallel, or equidistant; as GLHM.



XXXVI. In a Parallelogram ABCD, when a diameter AC, and two lines EF, HI parallel to the sides, cutting the diameter in one and the same point G, are drawn, so that the Parallelogram be divided by them into four parallelograms; those two DG, GB, through which the diameter passeth not, are called Complements; and the other two HE, FI, through which the diameter passeth, the Parallelograms standing about the diameter.

A Problem *is, when something is proposed to be done or effected.*

A Theoreme *is, when something is proposed to be demonstrated.*

A Corollary *is a consectary, or some consequent truth gained from a preceding demonstration.*

A Lemma *is the demonstration of some premise, whereby the proof of the thing in hand becomes the shorter.*

Postulates or Petitions.

1. FRom any point to any point to draw a right line.

2. To produce a right line finite, strait forth continually.

3. Upon any center, and at any distance, to describe a circle.

Axioms.

1. THings equal to the same third, are also equal one to the other.

As A=B=C. Therefore A=C. Or therefore all, A, B, C, are equal the one to the other.

Note, *When several quantities are joined the one to the other continually with this mark =, the first quantity is by virtue of this axiome equal to the last, and every one to every one: In which case we often abstain*

ſtain from citing the axiome, for brevity's ſake; altho' the force of the conſequence depend thereon.

2. If to equal things you add equal things, the wholes ſhall be equal.

3. If from equal things you take away equal things, the things remaining will be equal.

4. If to unequal things you add equal things, the wholes will be unequal.

5. If from unequal things you take away equal things, the remainders will be unequal.

6. Things which are double to the ſame third, or to equal things, are equal one to the other. Underſtand the ſame of triple, quadruple, *&c.*

7. Things which are half of one and the ſame thing, or of things equal, are equal the one to the other. Conceive the ſame of ſubtriple, ſubquadruple, *&c.*

8. Things which agree together, are equal one to the other.

The converſe of this axiome is true in right lines and angles, but not in figures, unleſs they be like.

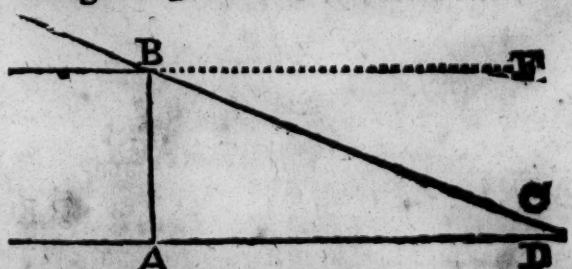
Moreover, magnitudes are ſaid to agree, when the parts of the one being applyed to the parts of the other, they fill up an equal or the ſame place.

9. Every whole is greater than its part.

10. Two right lines cannot have one and the ſame ſegment (or part) common to them both.

11. Two right lines meeting in the ſame point, if they be both produced, they ſhall neceſſarily cut one the other in that point.

12. All right angles are equal the one to the other.



13. If a right line BA falling on two right lines

AD, CB, make the internal angles on the same side, BAD, ABC, less than two right angles, those two right lines produced shall meet on that side, where the angles are less than two right angles.

14. Two right lines do not contain a space.

15. If to equal things, you add things unequal, the excess of the wholes shall be equal to the excess of the additions.

16. If to unequal things equal be added, the excess of the wholes shall be equal to the excess of those which were at first.

17. If from equal things, unequal things be taken away, the excess of the remainders shall be equal to the excess of the wholes.

18. If from things unequal, things equal be taken away, the excess of the remainders shall be equal to the excess of the wholes.

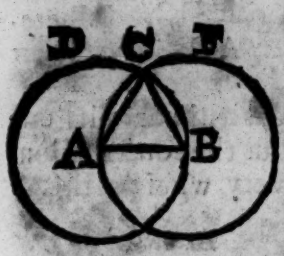
19. Every whole is equal to all its parts taken together.

20. If one whole be double to another, and that which is taken away from the first to that which is taken away from the second, the remainder of the first shall be double to the remainder of the second.

The Citations are to be understood in this manner; When you meet with two numbers, the first shews the Proposition, the second the Book; as by 4. 1. you are to understand the fourth Proposition of the first Book; and so of the rest. *Moreover, ax. denotes Axiome, post. Postulate, def. Definition, sch. Scholium, cor. Corollary.*

PRO-

PROPOSITION I.



Upon a finite right line given AB, to describe an equilateral triangle ACB.

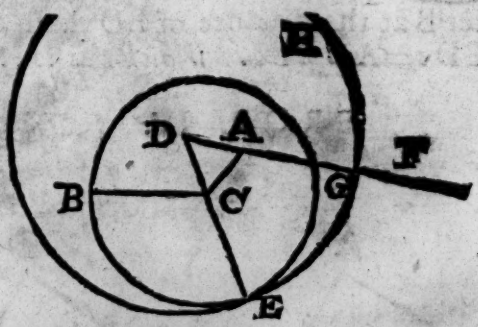
From the centers A and B, at the distance of AB, or BA, *a* describe two circles cutting each other in the point C; from whence *b* draw two right lines CA, CB. Then is AC *c* = AB *c* = BC *d* = AC. *e* Wherefore the Triangle ACB is equilateral. *Which was to be done.*

a 3. *post.*
b 1. *post.*
c 15. *def.*
d 1. *ax.*
e 23. *def.*

Scholium.

After the same manner upon the line AB may be described an Isosceles triangle, if the distances of the equal circles be taken greater or less than the line AB.

PROP. II.



At a point given A, to make a right line AG equal to a right line given BC.

From the center C, at the distance of CB, *a* describe the circle CBE. *b* join AC; upon which *c* raise the equilateral triangle ADC. *d* Produce DC to E. From the center D, at the distance of

a 3. *post.*
b 1. *post.*
c 1. 1.
d 2. *post.*

of

e 2. *post.* of DE, describe the circle DEH; and let DA e be produced to the point G in the circumference thereof. Then AG = CB.

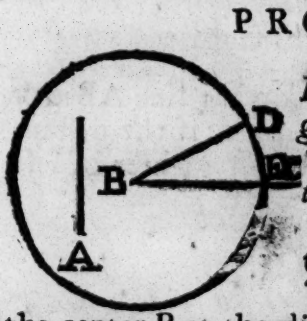
f 15. *def.* For DG *f* = DE, and DA *g* = DC. Where-
g *constr.* fore AG *h* = CE *k* = BC *l* = AG. *Which was*
h 3. *ax.* *to be done.*
k 15. *def.* The putting of the point A within or without
l 1. *ax.* the line BC varies the cases; but the construction, and the demonstration, are every where alike.

Schol.

The line AG might be taken with a pair of compasses; but the so doing answers to no Postulate, as *Proclus* well intimates.

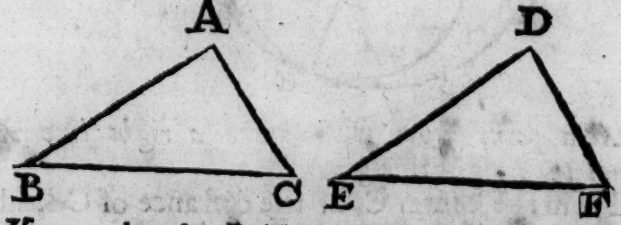
PROP. III.

Two right lines, A and BC, being given, from the greater BC to take away the right line BE equal to the lesser A.



a 2. 1. At the point B *a* draw the right line BD = A. The circle described from the center B at the distance of BD shall cut off
b 15. *def.* BE *b* = BD *c* = A *d* = BE. *Which was to be done.*
c *constr.*
d 1. *ax.*

PROP. IV.



If two triangles BAC, EDF, have two sides of the one BA, AC equal to two sides of the other ED, DF, each to its correspondent side (that is, BA = ED,

and

and AC=DF) and have the angle A equal to the angle D contained under the equal right lines; they ſhall have the baſe BC equal to the baſe EF; and the triangle BAC ſhall be equal to the triangle EDF; and the remaining angles B, C, ſhall be equal to the remaining angles E, F, each to each, under which the equal ſides are ſubtended.

If the point D be applied to the point A, and the right line DE plac'd upon the right line AB, the point E ſhall fall upon B, becauſe DE *a*=AB, alſo the right line DF ſhall fall upon AC, becauſe the angle A *a*=D. moreover the point F ſhall fall upon the point C, becauſe AC *a* = DF. Therefore the right lines EF, BC ſhall agree, becauſe they have the ſame Terms, and conſequently are equal. Wherefore the triangles BAC, DEF, and the angles B, E, as alſo the angles C, F, do agree, and are equal. *Which was to be Demonſtrated.*

a *hyp.* b 14. *ax.*

PROP. V.

The angles ABC, ACB, at the baſe of an Iſoſceles triangle ABC, are equal one to the other: And if the equal ſides AB, AC be produc'd, the angles CBD, BCE, under the baſe, ſhall be equal one to the other.

a Take AE=AD; and *b* join CD, and BE.

a 3. 1. b 1. *poſt.*

Becauſe, in the triangles ACD, ABE, are AB *c* = AC, and AE *d*=AD, and the angle A common to them both, *e* therefore is the angle ABE=ACD, and the angle AEB *e* = ADC, and the baſe BE *e*=CD; alſo EC *f* = DB. Therefore in the triangles BEC, BDC *g* ſhall be the angle ECB = DBC. *Which was to be Dem.* Alſo therefore the angle EBC= DCB. but the angle ABE *h*=ACD; therefore the angle ABC *k*=ACB. *Which was to be Dem.*

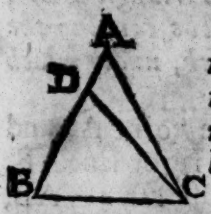
c *hyp.* d *conſtr.* e 4. 1. f 3. *ax.* g 4. 1. h *before.* k 3. *ax.*

Coroll.

Coroll.

Hence, every equilateral triangle is also equiangular.

PROP. VI.



If two angles ABC, ACB of a triangle ABC be equal the one to the other, the sides AC, AB subtended under the equal angles, shall also be equal one to the other.

If the sides be not equal, let one be bigger than the other, suppose BA ⊏ CA. *a* Make BD = CA, and *b* draw the line CD.

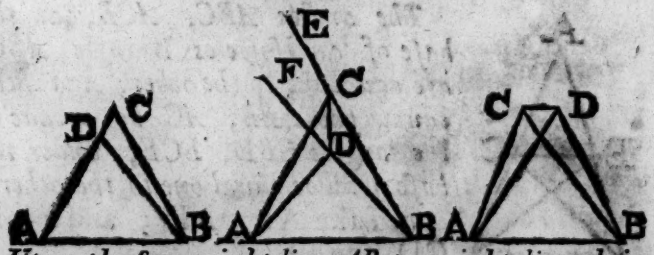
a 3. 1.
b 1. *post.*
c *suppos.*
d *hyp.*
e 4. 1.
f 9. *ax.*

In the triangles DBC, ACB, because BD *c* = CA, and the side BC is common, and the angle DBC *d* = ACB, the triangles DBC, ACB *e* shall be equal the one to the other, a part to the whole. *f Which is impossible.*

Coroll.

Hence, Every equiangular triangle is also equilateral.

PROP. VII.



Upon the same right line AB two right lines being drawn AC, BC, two other right lines equal to the former, AD, BD, each to each (viz. *AD = AC, and BD = BC*) *cannot be drawn from the same points A, B, on the same side C, to several points, as C and D, but only to C.*

1. *Case.* If the point D be set in the line AC,
a 9. *ax.* it is plain that AD is *a* not equal to AC.

2. *Case.* If the point D be placed within the triangle ACB, then draw the line CD, and produce BDF, and BCE. Now you would have AD = AC.

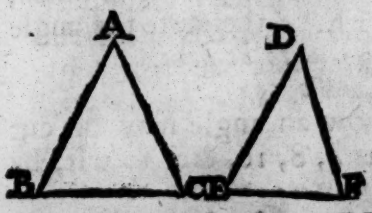
then

then the angle ADC *b*=ACD; as alſo, becauſe BD *c*= BC, the angle FDC = *b* ACD. therefore is the angle FDC ⊏ *d* ACD, that is, the angle FDC ⊏ ADC. *d Which is impoſſible.* b 5. 1. c *ſuppoſ.* d 9. *ax.*

3. *Caſe.* If D falls without the triangle ACB, let CD be joined.

Again the angle ACD *e* = ADC, and the angle BCD *e* = BDC. *f* Therefore the angle ACD ⊏ BDC, *viz.* the angle ADC ⊏ BDC. *Which is impoſſible.* Therefore, &c. e 5. 1. f 9. *ax.*

PROP. VIII.



If two triangles ABC, DEF, have two ſides AB, AC equal to two ſides DE, DF, each to each, and the baſe BC equal to the baſe EF, then the angles contained under the equal right lines ſhall be equal, viz. *A to D.*

Becauſe BC *a*=EF, if the baſe BC be laid on the baſe EF, *b* they will agree: therefore whereas AB *c*=DE, and AC=DF, the point A will fall on D (for it cannot fall on any other point, by the precedent propoſition) and ſo the ſides of the angles A and D are coincident; *d* wherefore thoſe angles are equal. *Which was to be Dem.* a *hyp.* b *ax.* 8. c *hyp.* d 8. *ax.*

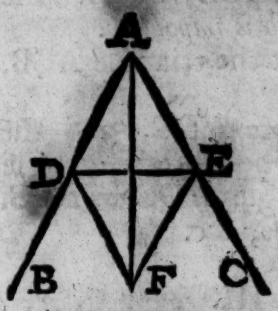
Coroll.

1. Hence, Triangles mutually equilateral, are alſo mutually *e* equiangular.

2. Triangles mutually equilateral, *e* are equal one to the other. e 4. 1.

PROP.

PROP. IX.



To biſect, or divide into two equal parts, a right-lined angle given BAC.

a 3. 1. *a* Take AD = to AE,

b 1. 1. and draw the line DE; upon which *b* make an equilateral triangle DFE. draw the right line AF; it ſhall biſect the angle.

c *conſtr.* For AD *c* = AE, and the ſide AF is common,

d 8. 1. and the baſe DF *c* = FE. *d* therefore the angle DAF = EAF. *Which was to be done.*

Coroll.

Hence it appears, how an angle may be cut into any equal parts, as 4, 8, 16, &c. to wit, by biſecting each part again.

The method of cutting angles into any equal parts required, by a Rule and Compaſs, is as yet unknown to Geometricians.

PROP. X.

To biſect a right line given AB.

a 1. 1. Upon the line given AB

b 9. 1. *a* erect an equilateral triangle ABC; and *b* biſect the angle C with the right line CD. That line ſhall alſo biſect the line given AB.

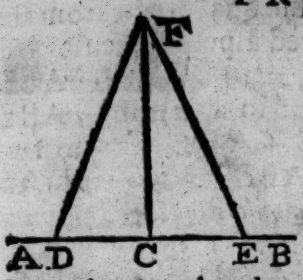
c *conſtr.* For AC *c* = BC, and the ſide CD is common,

d 4. 1. and the angle ACD *c* = BCD. therefore AD *d* = BD. *Which was to be done.*

The practice of this and the precedent Propoſition is eaſily ſhewn by the conſtruction of the 1 Prop. of this Book.

PROP.

PROP. XI.



From a point C in a right line given AB to erect a right line CF at right angles.

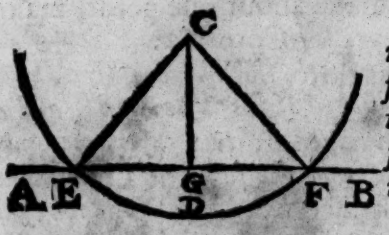
a Take on either side of the point given CD = CE. upon the right line DE *b* erect an equilateral triangle, draw the line FC, and it will be the perpendicular required.

For the triangles DFC, EFC are mutually *c* equilateral; *d* therefore the angle DCF = ECF. *e* therefore FC is perpendicular. *Which was to be done.*

a 3. 1.
b 1. 1.
c *constr.*
d 8. 1.
e 10. *def.*

The practice of this and the following is easily performed by the help of a square.

PROP. XII.



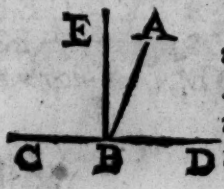
Upon an infinite right line given AB, from a point given that is not in it, to let fall a perpendicular right line CG.

From the center C *a* describe a circle cutting the right line given AB in the points E and F. Then bisect EF in G, and draw the right line CG, which will be the perpendicular required.

Let the lines CE, CF be drawn. The triangles EGC, FGC are mutually *c* equilateral. *d* therefore the angles EGC, FGC are equal, and by *e* consequence right. *e* Wherefore GC is a perpendicular. *Which was to be done.*

a 3. *post.*
b 10. 1.
c *constr.*
d 8. 1.
e 10. *def.*

PROP. XIII.



When a right line AB standing upon a right line CD maketh angles ABC, ABD; it maketh either two right angles, or two angles equal to two right.

If

a *def.* 10. If the angles ABC, ABD be equal, *a* then they make two right angles; if unequal, then from the point B *b* let there be erected a perpendicular BE. c 19. *ax.* Because the angle ABC *c* = to a right + ABE, d 3. *ax.* and the angles ABD *d* = to a right − ABE, e 2. *ax.* therefore shall be ABC + ABD *e* = to two right angles + ABE − ABE = two right angles. *Which was to be demonstrated.*

b 11. 1.

Corollaries.

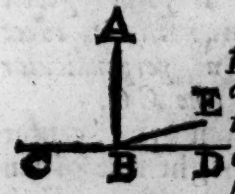
1. Hence, if one angle ABD be right, the other ABC is also right; if one acute, the other is obtuse, and so on the contrary.

2. If more right lines than one stand upon the same right line at the same point, the angles shall be equal to two right.

3. Two right lines cutting each other make angles equal to four right.

4. All the angles made about one point, make four right; as appears by Coroll. 2.

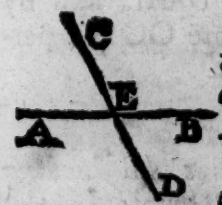
PROP. XIV.



If to any right line AB, and a point therein B, two right lines, not drawn from the same side, do make the angles ABC, ABD on each side equal to two right, the lines CB, BD shall make one strait line.

If you deny it, let CB, BE make one right line, a 13. 1. then shall be the angle ABC + ABE *a* = two right b *hyp.* angles *b* = ABC + ABD. *Which is absurd.* c 9. *ax.*

PROP. XV.

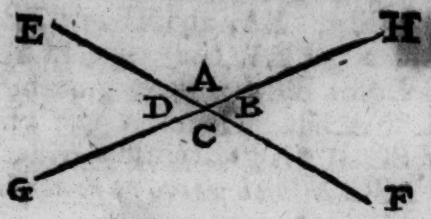


If two right lines AB, CD cut thro' one another, then are the two angles which are opposite, viz. CEB, AED, equal one to the other.

For the angle AEC + CEB *a* = to two right angles = AEC a 13. 1. + AED; *b* therefore CEB = AED. *Which was* b 3. *ax.* *to be demonstrated.*

Schol.

Schol.

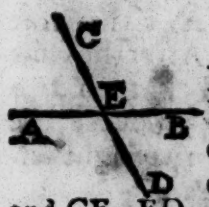


If to any right line GH, and in it a point A, two right lines being drawn EA, AF, and not taken on the ſame ſide, make the vertical (or oppoſite) angles D and B equal, thoſe right lines EA, AF, do meet directly and make one ſtrait line.

For two right angles are *a* equal to the angle D + A *a* = B + A. *b* therefore, EA, AF, are in a ſtrait line. *Which was to be demonſtrated.*

a 13. 1.
b 14. 1.

Schol.

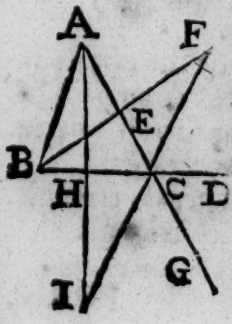


If four right lines EA, EB, EC, ED, proceeding from one point E, make the angles vertically oppoſite equal the one to the other, each two lines, AE, EB, and CE, ED, are placed in one ſtrait line.

For becauſe the angle AEC + AED + CEB + DEB *a* = to 4 right angles, therefore the angle AEC + AED *b* = CEB + DEB = to two right angles, *c* therefore CED and AEB are ſtrait lines. *Which was to be demonſtrated.*

a 4. *c.* 13. 1.
b *hyp. and* 2. *ax.*
c 14. 1.

PROP. XVI.



One ſide B C of any triangle A B C being produc'd, the outward angle A C D will be greater than either of the inward and oppoſite angles C A B, C B A.

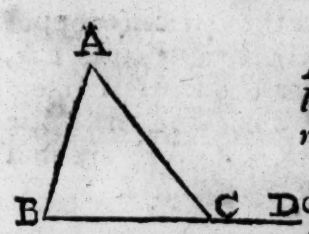
Let the right lines AH, BE *a* biſect the ſides A C, B C; *b* produce EF = BE, and HI, a 10. 1. & 1. *poſt.*

b = b 3. 1.

b = AH. and join FC, and IC; and produce ACG.

c *constr.* Because CE c = EA, and EF c = EB, and the
d 15. 1. angle FEC d = BEA, the angle ECF e shall be
e 4. 1. equal to EAB. By the like argument is the an-
f 15. 1. gle ICH = ABH. Therefore the whole angle
g 9. *ax.* ACD (*f* BCG) g is greater than either the angle CAB or ABC. *Which was to be demonstrated.*

PROP. XVII.



Two angles of any triangle ABC, which way soever they be taken, are less than two right angles.

Let the side BC be produced. Because the angle ACD + ACB a = 2 right
a 13. 1. angles, and the angle ACD b ⊏ A, c therefore
b 16. 1. A + ACB ⊐ then two right angles. After the same
c 4. *ax.* manner is the angle B + ACB ⊐ then two right. Lastly, the side AB being produced, the angle A + B will be also less than two right angles. *Which was to be demonstrated.*

Coroll.

1. Hence it follows that in every triangle, wherein one angle is either right or obtuse, the two others are acute angles.

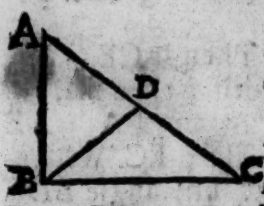
2. If a right line AE make unequal angles with another right line D, one acute AED, the other obtuse AEC, a perpendicular AD let fall from any point A to the other line CD, shall fall on that side the acute is of.

For if AC, drawn on the side of the obtuse angle, be a perpendicular, then in the triangle AEC
* 17. 1. shall AEC + ACE be * greater than two right angles. *Which is contrary to the precedent Prop.*

3. All the angles of an equilateral triangle, and the two angles of an Isosceles triangle that are upon the base, are acute.

PROP.

PROP. XVIII.

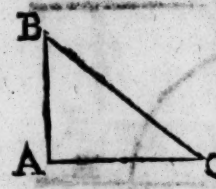


The greateſt ſide AC of every triangle ABC ſubtends the greateſt angle ABC.

From AC *a* take away AD = AB, and join BD. *b* Therefore is the angle ADB = ABD. But ADB *c* ⊏ C; therefore is ABD ⊏ C; *d* therefore the whole angle ABC ⊏ C. After the ſame manner, ſhall be ABC ⊏ A. *Which was to be dem.*

a 3. 1. b 5. 1. c 16. 1. d 9. *ax.*

PROP. XIX.

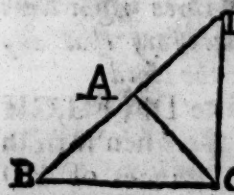


In every triangle ABC, under the greateſt angle A is ſubtended the greateſt ſide BC.

For if AB be ſuppoſed equal to BC, then will be the angle A *a* = C, which is contrary to the Hypotheſis: and if AB ⊏ BC, then ſhall be the angle C *b* ⊏ A, which is againſt the Hypotheſis. Wherefore rather BC ⊏ AB; and after the ſame manner BC ⊏ AC. *Which was to be dem.*

a 5. 1. b 18. 1.

PROP. XX.

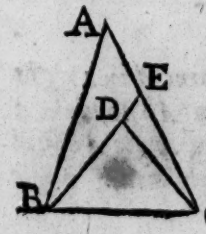


Of every triangle ABC two ſides BA, AC, any way taken, are greater than the ſide that remains BC.

Produce the line BA, *a* and take AD = AC, and draw the line DC, *b* then ſhall the angle D be equal to ACD, *c* therefore is the whole angle BCD ⊏ D; *d* therefore BD (*e* BA + AC) ⊏ BC. *Which was to be demonſtrated.*

a 3. 1. b 5. 1. c 9. *ax.* d 19. 1. e *conſtr.* & 2. *ax.*

PROP. XXI.

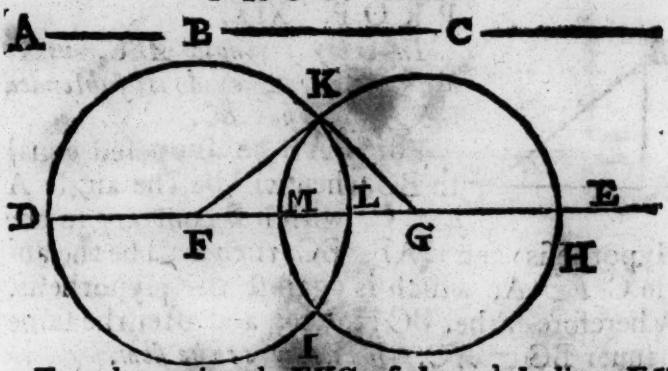


If from the utmoſt points of one ſide BC of a triangle A B C two right lines B D, C D be drawn to any point within the triangle, then are both thoſe two lines ſhorter than the two other ſides of

the triangle, BA, CA; but do contain a greater angle, BDC.

Let BD be produced to E. Then is CE + ED
a = CD, and BD common to both *b* then shall be
a 20. 1. BD DE + EC = CD + BD. Again, BA AE
b 4. *ax.* *a* = BE. *b* therefore BA AC = BE EC. Where-
fore 1. BA + AC = BD + DC. 2. The angle BDC
c 16. 1. *c* = DEC *c* A. Therefore the angle BDC = A.
Which was to be demonstrated.

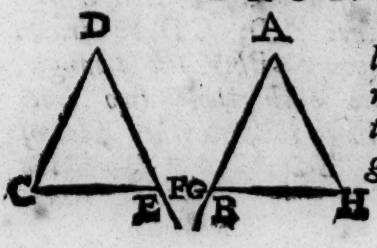
PROP. XXII.



To make a triangle FKG of three right lines FK, FG, GK, which shall be equal to three right lines given A, B, C. Of which it is necessary that any two taken together be longer than the third.

From the infinite line DE *a* take DF, FG, GH
a 3. 1. equal to the lines given, A, B, C. Then if from
the *b* centers F and G by the distances of FD
b 3. *post.* and GH, two circles be drawn cutting each
other in K, and the right lines KF, KG be join-
c 15. *def.* ed, the triangle FKG shall be made, *c* whose
sides FK, FG, GK are equal to the three lines
d 1. *ax.* DF, FG, GH *d* that is to the three lines given
A, B, C. *Which was to be done.*

PROP. XXIII.

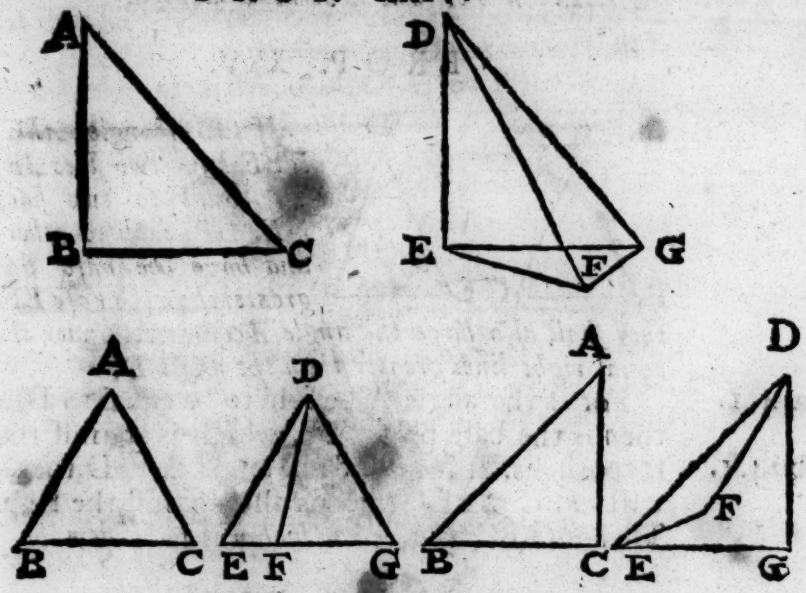


At a point A in a right line given AB, to make a right-lined angle A equal to a right-lined angle given D.

a 1. *post.* Draw the right line

line CF cutting the sides of the angle given any ways; *b* make AG = CD; upon AG *c* raise a trian- b 3. 1.
gle equilateral to the former CDF, so that AH be c 21. 1.
equal to DF, and GH to CF. then shall you have
the angle A *d* = D. *Which was to be done.* d 8. 1.

PROP. XXIV.



If two triangles ABC, DEF have two sides of the one triangle AB, AC equal to two sides of the other triangle DE, DF, each to other, and have the angle A greater than the angle EDF contained under the equal right lines, they shall also have the base BC greater than the base EF.

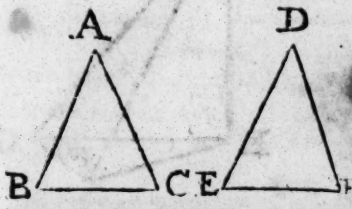
a Let the angle EDG be made equal to A, a 23. 1.
and the side DG *b* = DF *c* = AC; and let EG, b 3. 1.
and FG be joined. c *hyp.*

1. *Case.* If EG fall above EF; because AB *d* = d *hyp.*
DE, and AC *e* = DG, and the angle A *e* = EDG, e *constr.*
f therefore is BC = EG. But because DF *e* = DG, f 4. 1.
g therefore is the angle DFG = DGF; *h* therefore g 5. 1.
is the angle DFG ⊐ FFG, and by consequence h 9. *ax.*
the angle EFG *k* ⊐ EGF, *k* wherefore EG (BC) k 19. 1.
⊐ EF.

2. *Case.* If the base EF falls in the same place
l 9. *ax.* with EG, *l* it is evident that EG (BC) ⊏ EF.

3. *Case.* If EG fall below EF, then because DG
m 21. 1. + GE *m* ⊏ DF + FE, if from both DG, DF be taken away, which are equal, EG (BC) remains
n 5. *ax.* *n* ⊏ EF. *Which was to be demonstrated.*

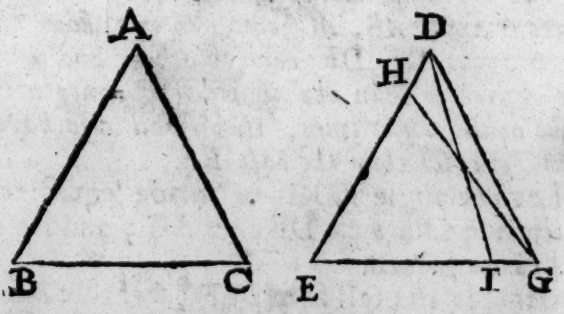
PROP. XXV.



If two triangles ABC, DEF have two sides AB, AC equal to two sides DE, DF, each to other, and have the base BC greater than the base EF, they shall also have the angle A contained under the equal right lines greater than the angle D.

a 4. 1. For if the angle A be said to be equal to D, *a* then is the base BC = EF, which is against the
b 24. 1. Hypothesis. If it be said the angle A ⊐ D, then *b* will be BC ⊐ EF, which is also against the Hyp. Therefore BC ⊏ EF. *Which was to be dem.*

PROP. XXVI.



If two triangles BAC, EDG have two angles of the one B, C equal to the two angles of the other E, DGE, each to his correspondent angle, and have also one side of the one equal to one side of the other, either that side which lyeth betwixt the equal angles, or that which is subtended under one of the equal angles; the other sides also

alſo of the one ſhall be equal to the other ſides of the other each to his correſpondent ſide, and the other angle of the one ſhall be equal to the other angle of the other.

1. *Hyp.* Let BC be equal to EG, which are the ſides that lie between the equal angles. Then I ſay BA=ED, and AC=DG, and the angle A=EDG. For if it be ſaid that ED⌐BA, then *a* let EH be made equal to BA, and let the line GH be drawn.

Becauſe AB *b*=HE, and BC *c* = EG, and the angle B *c*=E, therefore ſhall be the angle EGH *d* = C *e* = DGE. *f Which is abſurd.* After the ſame manner let AC be equal to DG, *d* then will the angle A be equal to EDG.

a 3. 1.
b *ſuppoſ.*
c *hyp.*
d 4. 1.
e *hyp.*
f 9. *ax.*

2. *Hyp.* Let AB be equal to ED. Then I ſay BC=EG, and AC=DG, and the angle A=EDG. For if EG be greater than BC, make EI=BC, and join the line DI. Now becauſe AB *g*=ED, and BC *h* = EI, and the angle G *g* = E; therefore will be the angle EID *k*=C *l*=EGD. *m Which is abſurd.* Therefore is BC=EG, and ſo as before AC=DG, and the angle A=EDG. *Which was to be demonſtrated.*

g *hyp.*
h *ſuppoſ.*
k 4. 1.
l *hyp.*
m 16. 1.

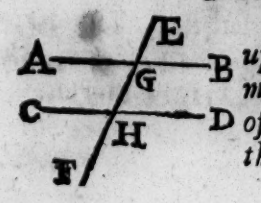
PROP. XXVII.

If a right line EF falling upon two right lines AB, CD, make the alternate angles AEF, DFE, equal the one to the other, then are the right lines AB, CD parallel.

If AB, CD be ſaid not to be parallel, produce them till they meet in G. which being ſuppoſed, the outward angle AEF will be *a* greater than the inward angle DFE, to which it was equal by Hypotheſis. *Which is repugnant.*

a 16. 1.

PROP. XXVIII.



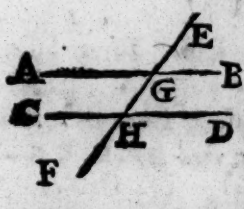
If a right line EF falling upon two right lines, AB, CD make the outward angle AGE of the one line equal to CHG the inward and oppoſite angle

of the other on the same side, or make the inward angles on the same side AGH, CHG equal to two right angles, then are the right lines AB, CD parallel.

Hyp. 1. Because by Hypothesis the angle AGE
a 25. 1. = CHG, *a* therefore are BGH, CHG alternate
b 27. 1. angles and equal: And *b* therefore are AB and CD parallel.

Hyp. 2. Because by Hypothesis the angle AGH
a 13. 1. + CHG = to two right, *a* = AGH + BGH,
b 3. *ax.* *b* thence is CHG = BGH; and *c* therefore AB,
c 27. 1. CD are parallel. *Which was to be demonstrated.*

PROP. XXIX.



If a right line EF fall upon two parallels, AB, CD, it will both make the alternate angles DHG, AGH equal each to other, and the outward angle BGE equal to the inward and opposite angle on the same side DHE, as also the inward angles on the same side AGH, CHG equal to two right angles.

It is evident that AGH + CHG = 2 right an-
a 13. *ax.* gles; *a* otherwise AB, CD would not be parallel,
b 13. 1. which is contrary to the Hyp. But moreover
c 13. *ax.* the angle DHG + CHG *b* = 2 right; therefore is
d 15. 1. DHG + AHG *d* = BGE. *Which was to be dem.*

Coroll.

Hence it follows that every parallelogram AC having one angle right A, the rest are also right.

a 29. 1. For A + B *a* = 2 right angles. Therefore
b 3. *ax.* whereas A is right, *b* must B be also right. By the same argument are C and D right angles.

PROP.

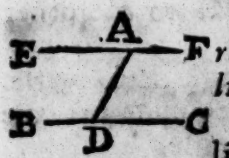
PROP. XXX.

Right lines (AB, CD) parallel to one and the same right line EF, are also parallel the one to the other.

Let GI cut the three right lines given any ways, then because AB, EF are parallel, will be the angle AGI *a* = EHI. also because CD and EF are parallel, will be the angle EHI *a* = DIG. *b* Therefore the angle AGI = DIG. *c* whence AB and CD are parallel. *Which was to be dem.*

a 29. 1.
b 1. *ax.*
c 27. 1.

PROP. XXXI.

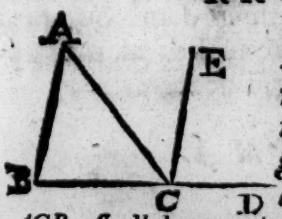


From a point given A to draw a right line AE parallel to a right line given BC.

From the point A draw a right line AD to any point of the given right line; with which at the point thereof *a* A make an angle DAE = ADC. *b* then will AE and BC be parallel. *Which was to be done.*

a 23. 1.
b 27. 1.

RROP XXXII.



Of any triangle ABC one side BC, being drawn out, the outward angle ACD shall be equal to the two inward opposite angles A, B, and the three inward angles of the triangle, A, B, ACB, shall be equal to two right angles.

From C *a* draw CE parallel to BA. Then is the angle A *b* = ACE, and the angle B *b* = ECD. Therefore A + B *c* ACE + ECD *d* = ACD. *Which was to be demonstrated.*

a 31. 1.
b 29. 1.
c 2. *ax.*
d 19. *ax.*

I affirm ACD + ACB *e* = two right angles; *f* therefore A + B + ACB = 2 right angles. *Which was to be demonstrated.*

e 13. 1.
f 1. *ax.*

Coroll.

1. The three angles of any triangle taken together

ther are equal to three angles of any other triangle taken together. From whence it follows,

2. That if in one triangle, two angles (taken ſeverally, or together) be equal to two angles of another triangle (taken ſeverally, or together) then is the remaining angle of the one equal to the remaining angle of the other. In like manner, if two triangles have one angle of the one equal to one of the other, then is the ſum of the remaining angles of the one triangle equal to the ſum of the remaining angles of the other.

3. If one angle in a triangle be right, the other two are equal to a right. Likewiſe, that angle in a triangle which is equal to the other two, is it ſelf a right angle.

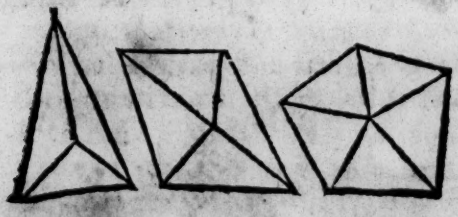
4. When in an Iſoſceles the angle made by the equal ſides is right, the other two upon the baſe are each of them half a right angle.

5. An angle of an equilateral triangle makes two third parts of a right angle. For $\frac{1}{3}$ of two right angles is equal to $\frac{2}{3}$ of one.

Schol.

By the help of this Propoſition you may know how many right angles the inward and outward angles of a right-lined figure make; as may appear by theſe two following Theorems.

THEOREM I.



All the angles of a right-lined figure do together make twice as many right angles, bating four, as there are ſides of the figure.

From any point within the figure, let right lines

lines be drawn all thro' the angles of the figure, which ſhall reſolve the figure into as many triangles as there are ſides of the figure. Wherefore, whereas every triangle affords two right angles, all the triangles taken together will make up twice as many right angles as there are ſides. But the angles about the ſaid point within the figure make up four right; therefore, if from the angles of all the triangles you take away all the angles which are about the ſaid point, the remaining angles, which make up the angles of the figure, will make twice as many right angles, bating four, as there are ſides of the figure. *Which was to be dem.*

Coroll.

Hence, All right-lined figures of the ſame ſpecies have the ſums of their angles equal.

THEOREM II.

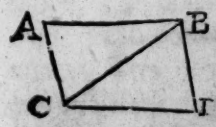
All the outward angles of any right-lined figure, taken together, make up four right angles.

For all the ſeveral inward angles of a figure with the ſeveral outward angles of the ſame make two right angles; therefore all the inward angles, together with the outward, make twice as many right angles as there are ſides of the figure; but (as it was now ſhewn) all the inward angles with four right, make twice as many right as there are ſides of the figure; therefore the outward angles are equal to four right angles. *Which was to be dem.*

Coroll.

All right-lined figures of whatſoever ſpecies have the ſums of their outward angles equal.

PROP. XXXIII.



If two equal and parallel lines AB, CD be joined together with two other right lines AC, BD, then are thoſe lines alſo equal and parallel.

Draw a line from C to B. Now becauſe AB and CD are parallel, and the angle ABC *a* = BCD; and alſo by Hypotheſis AB = CD, and the ſide CB a 29. 1.

com-

b 4. 1. c 27. 1. common, therefore is AC *b* = BD, and the angle ACB *b* = BD . *c* whence also AC, BD are parallel.

PROP. XXXIV.

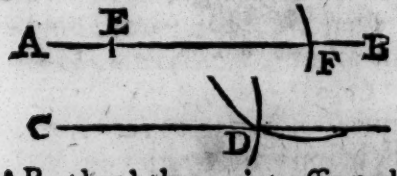
In parallelograms, as ABCD, the opposite sides AB, CD, and AC, BD, are equal each to other; and the opposite angles A, D, and ABD, ACD are also equal; and the diameter BC bisects the same.

a *hyp.* b 29. 1. c 2. *ax.* d 26. 1. Because AB, CD *a* are parallel, *b* therefore is the angle ABC = BCD. Also because AC, BD are *a* parallel, *b* therefore is the angle ACB = CBD; *c* therefore the whole angle ACD = ABD. After the same manner is A D. Moreover because the angles ABC, ACB lie at each end of the side CB, and are equal to BCD, CBD. *d* therefore is AC = BD, and AB *d* CD, and so the triangle ABC = CBD. *Which was to be dem.*

Schol.

Every four sided figure ABDC having the opposite sides equal, is a parallelogram.

a 27. 1. b 35. *def.* 1. For by 8. 1. the angle ABC = BCD; *a* wherefore AB, CD are parallel. In like manner is the angle BCA = CBD; *a* wherefore AC, BD are also parallel. *b* Therefore ABCD is a parallelogram. *Which was to be demonstrated.*

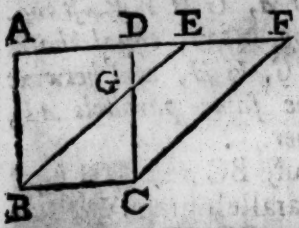


From hence may be learned how to draw a parallel CD to a right line given AB, thro' the point assigned C.

Take in the line AB any point, as E. From the centers E and C at any distance draw two equal circles EF, CD. From the center F by the space of EC draw a circle FD, which shall cut the former circle CD in the point D. Then shall the line drawn CD be parallel to AB. for as it was before demonstrated, CEFD is a parallelogram.

PROP.

PROP. XXXV.



Parallelograms, BCDA, BCEF, which ſtand upon the ſame baſe BC, and between the ſame parallels AF, BC, are equal one to the other.

For AD *a* = BC *a* = EF, add DE common to both, *b* then is AE = DF. But alſo AB *a* = DC, and the angle A *c* = CDF. *d* Therefore is the triangle ABE = DCF. take away DGE common to both triangles, *e* then is the Trapezium ABGD = EGCF. add BGC common to both, *f* then is the parallelogram ABCD = EBCF. *Which was to be dem.*

a 34. 1. b 2. *ax.* c 29. 1. d 4. 1. e 3. *ax.* f 2. *ax.*

The demonſtration of any other caſes is not unlike, but much more plain and eaſy.

Schol.

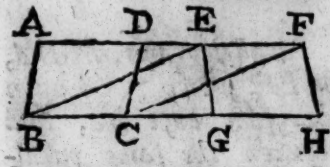
If the ſide AB of a right-angled parallelogram ABCD be conceived to be carried along perpendicularly thro' the whole line BC, or BC thro' the whole line AB, the Area or content of the Rectangle ABCD ſhall be produc'd by that motion. Hence a rectangle is ſaid to be made by the drawing or multiplication of two contiguous ſides. For example's ſake; let AB be ſuppoſed four foot, and BC be three: draw 3 into 4, there will be produced 12 ſquare feet for the Area of the Rectangle.

This being ſuppoſed, the dimenſion of any parallelogram (*EBCF) is found out by this Theorem. For the Area thereof is produced from the altitude BA drawn into the baſe BC. So the Area of the parallelogram AC = EBCF, is made by the drawing of BA into BC, therefore, *&c.*

* *See the figure of* Prop. 35.

PROP.

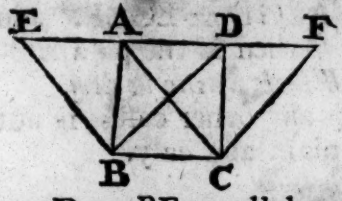
PROP. XXXVI.



Parallelograms, BCDA, GHFE, standing upon equal bases BC, GH, and betwixt the same parallels AF, BH, are equal to the other.

a *hyp.* Draw BE and CE. Because BC *a* = GH *b* =
b 34. 1. EF, *c* therefore is BCEF a parallelogram. Whence
c 33. 1. the parallelogram BCDA *d* = BCFE *d* = GHFE.
d 35. 1. *Which was to be dem.*

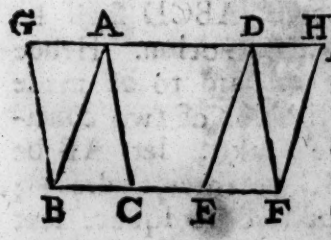
PROP. XXXVII.



Triangles, BCA, BCD, standing upon the same base BC, and between the same parallels BC, EF, are equal one to the other.

a 31. 1. *a* Draw BE parallel to CA, *a* and CF parallel
b 34. 1. to BD. Then is the triangle BCA *b* = ½ of the
c 35. 1. *and* 7. *ax.* parallelogram BCAE *c* = ½ BDFC *b* = BCD.
Which was to be dem.

PROP. XXXVIII.



Triangles, BCA, EFD, set upon equal bases BC, EF, and between the same parallels GH, BF, are equal the one to the other.

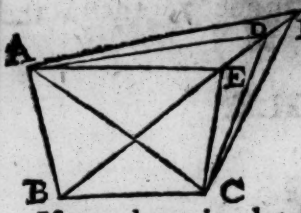
Draw BG parallel to CA and FH parallel to ED. Then is the trian-
a 34. 1. gle BCA *a* = ½ Pgr. BCAG *b* = ½ EDHF *c*
b 36. 1. *and* 7. *ax.* EFD. *Which was to be dem.*

Schol.

c 34. 1. If the base BC be greater than EF, then is the triangle BAC ⊏ EDF, and so on the contrary.

PROP.

PROP. XXXIX.

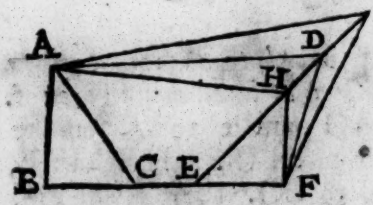


Equal triangles BCA, BCD, ſtanding on the ſame baſe BC, and on the ſame ſide, they are alſo between the ſame parallels AD, BC.

If you deny it, let another line AF be parallel to BC; and let CF be drawn. Then is the triangle CBF *a*=CBA *b*=CBD. *c Which is abſurd.*

a 37. 1.
b *hyp.*
c 9. *ax.*

PROP. XL.

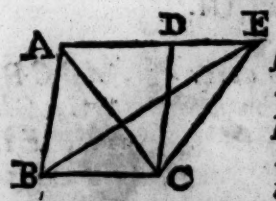


Equal triangles BCA, EFD, ſtanding upon equal baſes BC, EF, and on the ſame ſide, they are betwixt the ſame parallels.

If you deny it, let another line AH be parallel to BF, and let FH be drawn. Then is the triangle EFH *a*=BCA *b*=EFD. *c Which is abſurd.*

a 38. 1.
b *hyp.*
c 9. *ax.*

PROP. XLI.



If a Pgr. ABCD have the ſame baſe BC with the triangle BCE, and be between the ſame parallels AE, BC then is the Pgr. ABCD double to the triangle BCE.

Let the line AC be drawn. Then is the triangle BCA *a* = BCE. therefore is the Pgr. ABCD. *b* = 2 BCA *c* = 2 BCE. *Which was to be dem.*

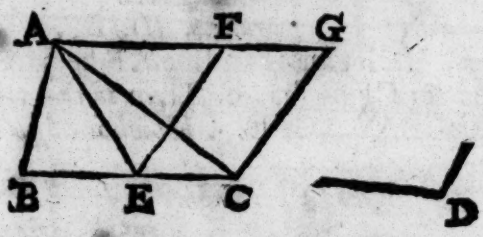
a 37. 1.
b 34. 1.
c 6. *ax.*

Schol.

From hence may the Area of any triangle BCE be found, for whereas the Area of the Pgr. ABCD is produced by the altitude drawn into the baſe, therefore ſhall the Area of a triangle be produced by half the altitude drawn into the baſe, or half the

the base drawn into the altitude, as if so be the base BC be 8, and the altitude 7, then is the Area of the triangle BCE, 28.

PROP. XLII.

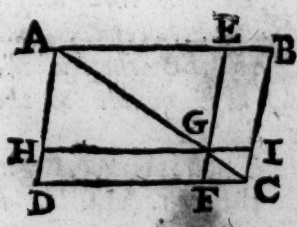


To make a Pgr. ECGF equal to a triangle given ABC in an angle equal to a right-lined angle given A.

a 31. 1. b 23. 1. c 10. 1. Through A *a* draw AG parallel to BC *b* make the angle BCG = D. *c* bisect the base BC in E, and draw EF parallel to CG. then is the problem resolved.

d 31. 1. e 41. 1. For AE being drawn, the angle ECG is equal to D by construction, and the triangle BAC *d* = 2 AEC *e* = Pgr. ECGF. *Which was to be done.*

PROP. XLIII.

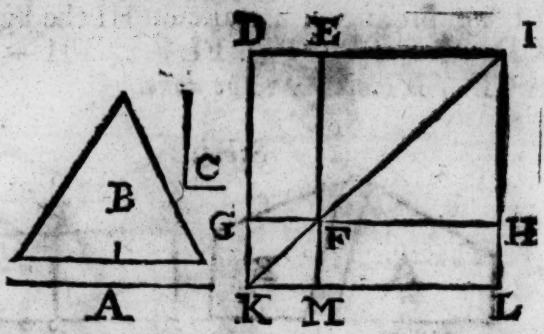


In every Pgr. AB CD, the complements DG, GB of those Pgrs HE, FI, which stand about the diameter, are equal one to the other.

a 34. 1. b 3. *ax.* For the triangle ACD *a* = ACB, and the triangle AGH *a* = AGE, and the triangle GCF *a* = GCI. *b* Therefore the Pgr. DG = BG. *Which was to be dem.*

PROP.

PROP. XLIV.



Upon a right line given A, to make a Parallelogram FL at a right-lined angle given C, equal to a triangle given B.

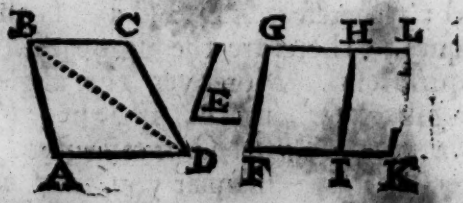
a Make a Pgr. FD = to the triangle B, so that the angle GFE may be equal to C. Produce GF till FH be equal to the line given A. thro' H *b* draw IL parallel to EF, which let DE produced meet in the point I. let DG drawn forth meet with a right line drawn from I in the point K. thro' K *b* draw KL parallel to GH, with which let EF drawn out meet at M, and IH at L. Then shall FL be the Pgr. required.

a 42. 1.
b 31. 1.

For the Pgr. FL *c* = FD = B, *d* and the angle MFH = GFE = C. *Which was to be done.*

c 43. 1.
d 15. 1.

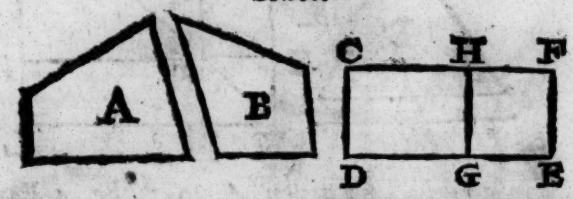
PROP. XLV.



Upon a right line given FG to make a Pgr. FL equal to a right-lined figure given ABCD, at a right-lined angle given E.

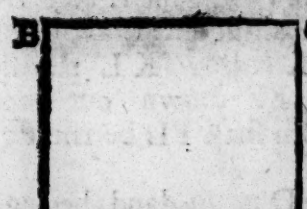
Resolve the right-lined figure given into tri-
a 44. 1. angles BAD, BCD. then *a* make a Pgr. FH =
BAD, so that the angle F may be equal to E.
FI being produced, *a* make on HI the Pgr. IL=
b 19. *ax.* BCD. Then is the Pgr. FL *b* = FH + IL *c* =
c *constr.* ABCD. *Which was to be done.*

Schol.



Hence is easily found the excess, HE, whereby any right-lined figure, A, exceeds a less right-lined figure, B; namely, if to some right line, CD, both be applied, Pgr. DF=A, and DH=B.

PROP. XLVI.



Upon a right line given AD to describe a square AC.

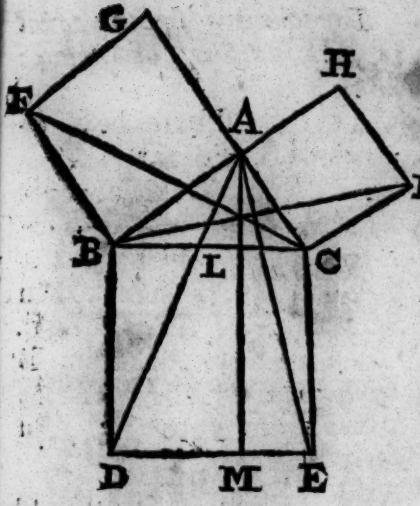
a 11. 3. *a* Erect two perpendiculars AB, DC, *b* equal
b 3. 1. to the line given AD; then join BC, and the thing required is done.

c *constr.* For, whereas the angle A + D *c* = 2 right, *d*
d 28. 1. therefore are AB, DC parallel. But they are also
e *constr.* *e* equal; *f* therefore AD, BC are both parallel
f 34. 1. and equal; therefore the figure AC is a Pgr.
g *sch.* 29. 1. and equilateral. Moreover the angles are all
h 29. *def.* right, *g* because one, A, is right; *h* therefore AC is a square. *Which was to be done.*

After the same manner you may easily describe a Rectangle contained under two right lines given.

PROP.

PROP. XLVII.



In right angled Triangles, BAC, the ſquare BE, which is made of the ſide BC that ſubtends the right angle BAC is equal to both the ſquares BG, CH, which are made of the ſides AB, AC containing the right angle.

Join AE, and AD; and draw AM parallel to CE.

Becauſe the angle DBC *a* = FBA, add the angle ABC common to them both; then is the angle ABD = FBC. Moreover AB *b* = FB, and BD *b* = BC; *c* therefore is the triangle ABD = FBC. But the Pgr. BM *d* = 2 ABD, and the Pgr. *d* BG = 2 FBC (for GAC is one right line by Hyp. and 14. 1.) *e* therefore is the Pgr. BM = BG. By the ſame way of argument is the Pgr. CM = CH. Therefore is the whole BE = BG + CH. *Which was to be dem.*

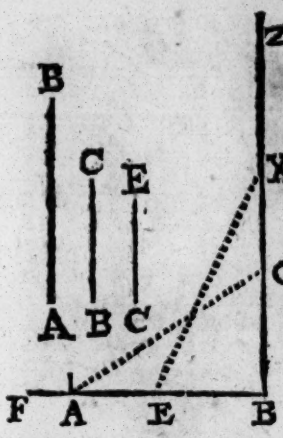
a 12. *ax.*
b 29. *def.*
c 4. 1.
d 41. 1.
e 6. *ax.*

Schol.

This moſt excellent and uſeful Theorem has deſerved the title of *Pythagoras*'s Theorem, becauſe he was the Inventer of it. By the help of which the addition and ſubſtraction of ſquares are performed; to which purpoſe ſerve the two following Problems.

PROBLEM I.

Andr. Tacq.



To make one square equal to any number of squares given.

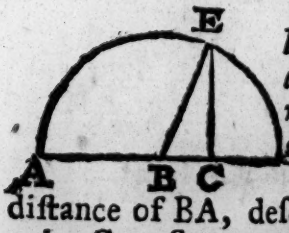
Let three squares be given whereof the sides are AB, BC, CE. *a* Make the right angle FBZ having the sides infinite; and on them transfer BA and BC; join AC: then is ACq *b* = ABq + BCq. Then transfer AC from B to X, and CE the third side given from B to E; join EX. *b* Then is EXq = EBq (CEq) + BXq (ACq) *c* = CEq + ABq + BCq. *Which was to be done.*

a 11. 1.

b 47. 1.

c 2. *ax.*

PROBLEM II.



Two unequal right lines being given AB, BC, to make a square equal to the difference of the two squares of the given lines, AB, BC.

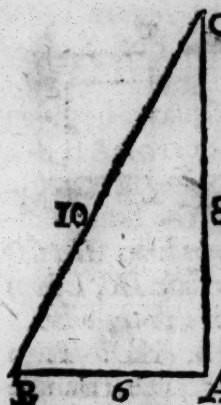
From the center B, at the distance of BA, describe a circle; and from the point C erect a perpendicular CE meeting with the circumference in E; and draw BE. *a* Then is BEq (BAq) = BCq + CEq. *b* Therefore BAq − BCq = CEq. *Which was to be done.*

a 47. 1.

b 3. *ax.*

PRO-

PROBLEM III.

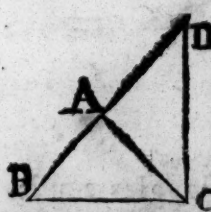


Any two sides of a right-angled triangle ABC being known, to find out the third.

Let the sides AB, AC encompassing the right angle, be, the one 6 foot, the other 8. Therefore, whereas ACq + ABq = 64 + 36 = 100 = BCq, thence is BC = $\sqrt{100}$ = 10. 47. 1.

Otherwise, let the sides AB, BC be known, the one six foot, the other 10. Therefore since BCq − ABq = 100 − 36 = 64 = ACq, thence is AC = $\sqrt{64}$ = 8. *Which was to be dem.* 47. 1.

PROP. XLVIII.



If the square made upon one side BC of a triangle be equal to the squares made on the other sides of the triangle AB, AC, then the angle BAC comprehended under those two other sides of the triangle AB, AC, is a right angle.

Draw to the point A in AC a perpendicular line DA = AB, and join CD.

Now is *a* CDq = ADq + ACq = ABq + ACq = BCq. * Therefore is CD = BC. And therefore the triangles CAB, CAD are mutually equilateral. Wherefore the angle CAB *b* = CAD *c* = a right angle. *Which was to be dem.*

a 47. 1.

* *See the following Theor.*

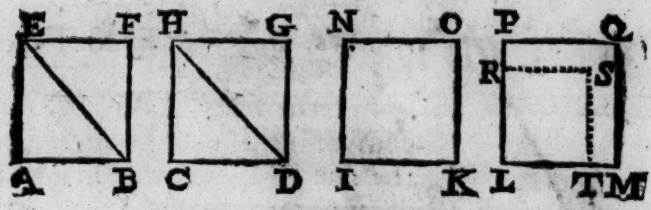
b 8. 1.

c *constr.*

Schol.

We assumed in the demonstration of the last Proposition, CD = BC, because CDq was equal to BCq: our assumption we prove by the following theorem.

THEOREME.



The squares AF, CG of equal right lines AB, CD are equal one to the other: And the sides IK, LM of equal squares NK, PM, are equal one to the other.

1. *Hyp.* Draw the diameters EB, HD. Then it is evident that AF is *a* equal to the triangle EAB twice taken, and *b* equal to the triangle HCD twice taken, and equal to *a* CG. *Which was to be done.*

a 34. 1.
b 4. 1. & 6. *ax.*

2. *Hyp.* If it may be, let LM be greater than IK. Make LT = IK, and let LS be *a* equal to LTq. Therefore is LS *b* = NK *c* = LQ. *d Which is absurd.*

a 46. 1.
b 1. *part*
c *hyp.*
d 9. *ax.*

Coroll.

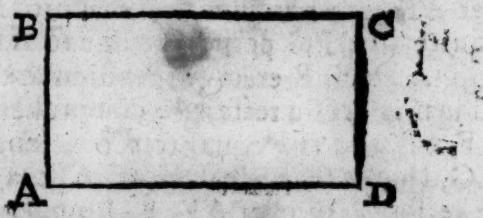
After the same manner any rectangles equilateral one to another, are demonstrated also to be equal.

The End of the first Book.

THE

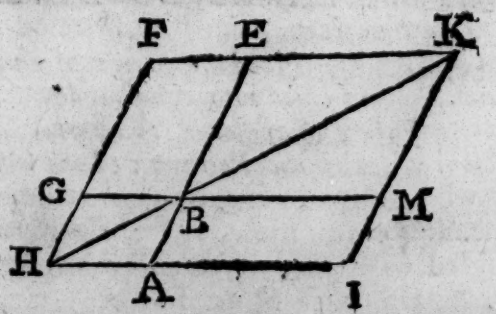
THE SECOND BOOK OF EUCLIDE's ELEMENTS.

Definitions.



I. EVery right-angled Parallelogram AB CD is ſaid to be contained under two right lines AB, AD comprehending a right angle.

Therefore when you meet with ſuch as theſe, the rectangle under BA, AD, or for ſhortneſs ſake the rectangle BAD, or BA x AD (or ZA, for Z x A) the rectangle meant is that which is contained under the right lines BA, AD ſet at right angles.



II. In every Pgr. FHIK, any one of thoſe parallelograms which are about the diameter, together with its two complements is called a Gnomon. *As the Pgr. FB + BI + GA (EHM) is a Gnomon; and likewiſe the Pgr. FB + BI + EM (GKA) is a Gnomon.*

PROP. I.

If two right lines AF, AB, be given, and one of them AB divided into as many parts or segments as you pleaſe; the rectangle comprehended under the two whole right lines AB, AF, ſhall be equal to all the rectangles contained under the whole line AF and the ſeveral ſegments, AD, DE, EB.

a 11. 1. *a* Set AF perpendicular to AB. Thro' F *a* draw an infinite line FG perpendicular to AF. From the points, D, E, B erect perpendiculars DH, EI, BG. Then is AG a rectangle comprehended under AF, AB, and is *b* equal to the rectangles AH, DI, EG, that is (becauſe DH, EI, AF *c* are equal) to the rectangles under AF, AD, under AF, DE, under AF, EB. *Which was to be dem.*

b 19. *ax.* 1.
c 34. 1.

Schol.

If two right lines given be both divided into how many parts ſoever, the product of the whole multiplied into itſelf ſhall be the ſame with that of the parts multiplied into themſelves.

For let Z be = A + B + C, and Y = D + E; then, becauſe DZ *a* = DA + DB + DC, and EZ *a* = EA + EB + EC, and YZ *a* = DZ + EZ, *b* ſhall ZY be = DA + DB + DC + EA + EB + EC. *Which was to be dem.*

a 1. 2.
b 2. *ax.*

From hence is underſtood the manner of multiplying compounded right lines into compounded. For you muſt take all the Rectangles of the parts, and they will preſent you with the Rectangle of the wholes.

But whenſoever in the multiplication of lines into themſelves you meet with theſe ſigns — intermingled with theſe +, you muſt alſo have particular regard to the ſigns. For of + multiplied into — ariſeth —; but of — into — ariſeth +, *ex. gr.* let + A be to be multiplied into B — C; then becauſe + A is not affirmed of all B, but only of a part of it, whereby it exceeds C, therefore AC muſt remain denied; ſo that the product will be

be AB − AC. Or thus; because B consists of the parts C and B − C, *thence AB = AC + A × B − C. take away AC from either, then AB − AC = A × B − C. In like manner if — A be to be multiplied into B − C, then seeing by reason of the figure — that A is not denied of all B. but only of so much as it exceeds C, therefore AC must remain affirmed. whence the product will be — AB + AC. Or thus; because AB = AC + A × B − C; take away all throughout, and there will be — AB = AC − A × − B + C; add AC to either, and there will be — AB + AC = A × B − C. * 1. 3.

This being sufficiently understood, the nine following Propositions, and innumerable others of that kind, arising from the comparing of lines multiplied into themselves (which you may find done to your hand in *Vieta* and other Analytical Writers) are demonstrated with great facility, by reducing the matter for the most part to almost a simple work.

Furthermore, *it appears that the product of any magnitude multiplied into the parts of any number, is equal to the product of the same multiplied into the whole number: As 5 A + 7 A = 12 A, and 4 A × 5 A + 4 A × 7 A = 4 A × 12 A. Wherefore what is here deliver'd of the multiplying of right lines into themselves, the same may be understood of the multiplying of numbers into themselves, so that whatsoever is affirmed concerning lines in the nine following Theoreme, holds good also in numbers; seeing they all immediately depend and are deriv'd from this first. * 19. ax.

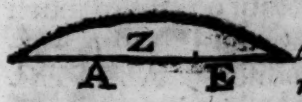
PROP. II.

If a right line Z be divided any-wise into two parts, the rectangles comprehended under the whole line Z and each of the segments A, E, are equal to the square made of the whole line Z.

I

I say that ZA + ZE = Zq. For take B = Z;
a 1. 2. *a* then is BA + BE = BZ, that is (because B = Z) ZA + ZE = Zq. *Which was to be dem.*

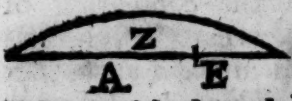
PROP. III.



If a right line Z be divided any-wise into two parts, the rectangle comprehended under the whole line Z and one of the segments E is equal to the rectangle made of the segments A, E, and the
a 1. 2. *square described on the said segment E.*

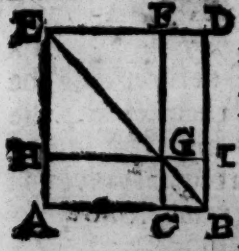
I say ZE = AE + Eq. *a* For EZ = EA + Eq.

PROP. IV.



If a right line Z be cut any-wise into two parts, the square made of the whole line
a 3. 2. *Z is equal both to the squares made of the segments A, E, and to twice a rectangle made of the parts A, E.*

b 2. 2. I say that Zq = Aq + Eq + 2 AE. For ZA *a* =
c 1. ax. Aq + AE, and ZE *a* = Eq + EA. Therefore whereas ZA + ZE *b* = Zq, thence is Zq = Aq + Eq + 2 AE. *Which was to be dem.*



Otherwise thus; Upon the right line AB make the square AD, and draw the diameter EB; through C, the point wherein the line AB is divided, draw the perpendicular CF; and thro' the point G draw HI parallel to AB.

Because the angle EHG = A is a right angle,
d 4. cor. 32. 1. and AEB is half a right, *e* therefore is the remaining angle HGE half a right angle. Therefore is HE *f* = HG *g* = EF *g* = AC, so that HF
e 32. 1. *h* is the square of the right line AC. After the
f 6. 1. same manner is CI proved to be CBq. Therefore
g 34. 1. AG, GD are rectangles under AC, CB. where-
h 29. def. fore the whole square AD *k* = ACq + CBq + 2
k 19. ax. 1. ACB. *Which was to be dem.*

Coroll.

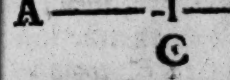
Coroll.

1. Hence it appears that the Parallelograms which are about the diameter of a ſquare are alſo ſquares themſelves.

2. That the diameter of any ſquare biſects its angles.

3. That if $A = \frac{1}{2} Z$, then $Zq = 4\,Aq$, and $Aq = \frac{1}{4} Zq$. As on the contrary, if it be ſo that $Zq = 4\,Aq$, then is $A = \frac{1}{2} Z$.

PROP. V.



If a right line AB be cut into equal parts AC, CB, and into unequal parts, AD, DB, the rectangle comprehended under the unequal parts AD, BD, together with the ſquare that is made of the intermediate part CD, is equal to the ſquare that is made of the half line CB.

I ſay that CBq = ADB + CDq.

For theſe are all equal.
- CBq.
- *a* CDq + CDB + DBq + CDB. — a 4. 2.
- CDq + *b* CBD (*c* AC × BD) + CDB. — b 3. 2. c *hyp.*
- CDq + *d* ADB. — d 1. 2.

This Theorem is ſomewhat differently expreſs'd, and more eaſily demonſtrated thus; *A Rectangle made of the ſum and the difference of two lines A, E, is equal to the difference of their ſquares.*

For if A + E be multiplied into A − E, there ariſeth Aq − AE + EA − Eq = Aq − Eq. *Which was to be dem.*

Schol.

If the line AB be divided otherwiſe, (*viz.*) nearer to the point of biſection, in E; Then is AEB ⊏ ADB.

For if AEB *a* = CBq − CEq. and ADB *a* = CBq − CDq. Therefore, whereas CDq ⊏ CEq, thence is AEB ⊏ ADB. *Which was to be dem.* — a 5. 2. *and* 3. *ax.*

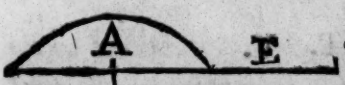
Coroll.

Coroll.

1. Hence is ADq + DBq ⊏ AEq + EBq. For
b 4. 2. ADq + DBq + 2 ADB *b* = ABq *b* = AEq + EBq + 2 AEB. Therefore because 2 AEB ⊏ 2 ADB. thence is ADq + DBq ⊏ AEq + EBq. *Which was to be dem.*

c 3. *ax.* 2. Hence is ADq + DBq − AEq + *c* + EBq = 2 AEB − 2 ADB.

PROP. VI.



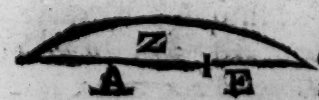
If a right line A be divided into two equal parts, and another right line E added to the same directly in one right line, then the rectangle comprehended under the whole and the line added, (viz. A + E) and the line added E, together with the square which is made of ½ the line A, is equal to the square of ½ A + E taken as one line.

a 4. & 3. Cor. 4. 2. I say that ¼ Aq (*a* Q: ½ A) + AE + Eq = Q. ½A + E. *a* For, Q ½ A + E = ¼ Aq + Eq + AE.

Coroll.

Hence it follows that if 3 right lines E, E + ½ A, E + A be in Arithmetical proportion, then the Rectangle contained under the extreme terms E, E + A, together with the square of the difference ½ A. is equal to the square of the middle term E + ½ A.

PROP. VII.



If a right line Z be divided any-wise into two parts, the square of the whole line Z together with the square made of one of the segments E, is equal to a double rectangle comprehended under the whole line Z and the said segment E, together with the square made of the other segment A.

a 4. 2. b 3. 2. I say [that Zq + Eq = 2 ZE + Aq. For Zq *a* = Aq + Eq + 2 AE. and 2 ZE *b* = 2 Eq + 2 AE. *Which was to be dem.*

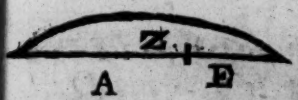
Coroll.

Coroll.

Hence it follows that the ſquare of the difference of any two lines Z, E, is equal to the ſquares of both the lines leſs by a double rectangle comprehended under the ſaid lines.

c For Zq + Eq − 2 ZE = Aq = Q: Z − E. c 7. 2: *and* 3. *ax.*

PROP. VIII.



If a right line Z be divided any-wiſe into two parts, the rectangle comprehended under the whole line Z and one of the ſegments E four times, together with the ſquare of the other ſegment A, is equal to the ſquare of the whole line Z and the ſegment E taken as one line Z + E.

I ſay that 4 ZE + Aq = Q: Z + E. For 2 ZE *a* = Zq + Eq − Aq. Therefore 4 ZE + Aq = Zq + Eq + 2 ZE *b* = Q: Z + E. *Which was to be dem.* a 7. 2. *and* 3. *ax.* b 4. 1.

PROP. IX.

If a right line AB be divided into equal parts AC, CB, and into unequal parts, AD, DB, then are the ſquares of the unequal parts AD, DB, together, double to the ſquare of the half line AC, and to the ſquare of the difference CD.

I ſay that ADq + DBq = 2 ACq + *a* CDq. For ADq + DBq *a* = ACq + CDq + 2 ACD + DBq. But 2 ACD (*b* 2 BCD) + DBq *c* = CBq (ACq) + CDq. *d* Therefore ADq + DBq = 2 ACq + 2 CDq. *Which was to be dem.* a 4. 1. b *hyp.* c 7. 2. d 2. *ax.*

This may be otherwiſe deliver'd and more eaſily demonſtrated thus; *The aggregate of the ſquares made of the ſum and the difference of two right lines A, E, is equal to the double of the ſquares made from thoſe lines.*

For Q: A + E *a* = Aq + Eq + 2 AE. and Q: A − E *b* = Aq + Eq − 2 AE. Theſe added together make 2 Aq + 2 Eq. *Which was to be dem.* a 4. 2. b *ſch.* 7. 2.

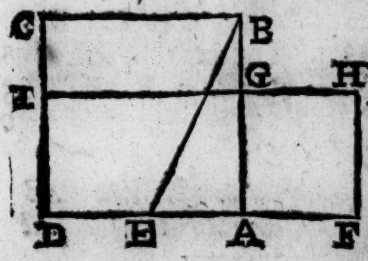
PROP.

PROP. X.

If a right line A be divided into two equal parts, and another line be added in a right line with the same, then is the square of the whole line, together with the added line, (as being one line,) together with the square of the added line E, double to the square of $\frac{1}{2}$ *A and the added line E, taken as one line.*

a 4. 2. I say that Eq+Q: A+E, *i. e. a* Aq+2 Eq+2
b *cor.* 4.2. AE = 2 Q: ^{1} A+2 Q: $\frac{1}{2}$ A+E. For 2 Q: $\frac{1}{2}$
c 4. 2. A *b* = $\frac{1}{2}$ Aq. And 2 Q: $\frac{1}{2}$ A+E *c* = $\frac{1}{2}$ Aq+ 2 Eq+2 AE. *Which was to be dem.*

PROP. XI.



To cut a right line given AB in point G, so that the rectangle comprehended under the whole line AB, and one of the segments BG shall be equal to the square that is made of the other segments AG.

a 46. 1. Upon AB *a* describe the square AC. *b* Bisect
b 10. 1. the side AD in E, and draw the line EB; from the line EA produced take EF = EB. On AF make the square AH. Then is AH = AB×BG.

For HG being drawn out to I; the rectan-
c 6. 2. gle DH + EAq *c* = EFq *d* = EBq *e* = BAq+EAq.
d *constr.* Therefore is DH *f* = BAq = to the square AC.
e 47. 1. Take away AI common to both, then remains
f 3. *ax.* the square AH = GC, that is, AGq = AB×BG
Which was to be done.

Schol.

This Proposition cannot be performed by
g 6. 13. numbers; * for there is no number that can be so divided, that the product of the whole into one part shall be equal to the square of the other part.

PROP.

PROP. XII.

In obtuſe-angled triangles, ABC, the ſquare that is made of the ſide AC ſubtending the obtuſe angle ABC is greater than the ſquares of the ſides BC, AB that contain the obtuſe angle ABC, by a double rectangle contained uuder one of the ſides BC, which are about the obtuſe angle ABC, on which ſide produced the perpendicular AD falls, and under the line BD, taken without the triangle from the point on which the perpendicular AD falls to the obtuſe angle ABC.

I ſay that ACq = CBq + ABq + 2 CB × BD.

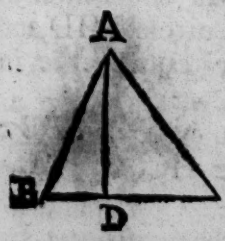
For theſe are all equal.
ACq.
a CDq + ADq. a 47. 1.
b CBq + 2 CBD + BDq + ADq. b 4. 1.
CBq + 2 CBD + *c* ABq. c 47. 1.

Scholium.

Hence, *the ſides of any obtuſe angled triangle ABC being known, the ſegment BD intercepted betwixt the perpendicular AD and the obtuſe angle ABC, as alſo the perpendicular it ſelf AD ſhall be eaſily found out.*

Thus. Let AC be 10, AB 7, CB 5. Then is ACq 100, ABq 49, CBq 25. And ABq + CBq = 74. Take that out of 100, then will 26 remain for 2 CBD. Wherefore CBD ſhall be 13; divide this by CB 5, there will 2 $\frac{3}{5}$ be found for BD. Whence AD will be found out by the 47. 1.

PROP. XIII.



In acute-angled triangles ABC, the ſquare made of the ſide AB ſubtending the acute angle ACB, is leſs than the ſquares made of the ſides AC, CB comprehending the acute angle ACB by a double rectangle contained under one of the ſides BC, which are about the acute angle ACB on which the perpendicular AD falls, and under the line DC taken within the triangle from the perpendicular AD to the acute angle ACB.

I say that ACq + BCq = ABq + 2 BCD.

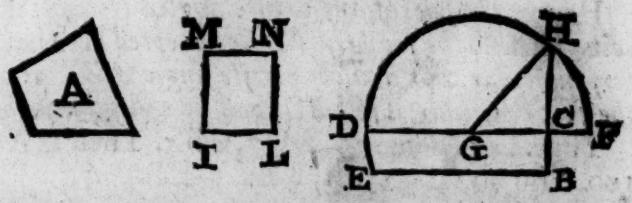
a 47. 1. For these are equal.
- ACq + BCq.
- *a* ADq + DCq + BCq.
- b 7. 2. *b* ADq + BDq + 2 BCD.
- c 47. 1. *c* ABq + 2 BCD.

Coroll.

Hence, *The sides of an acute-angled triangle ABC being known, you may find out the segment DC intercepted betwixt the perpendicular AD and the acute angle ACB, as also the perpendicular it self AD.*

Let AB be 13, AC 15, BC 14. Take ABq (169) from ACq + BCq, that is, from 225+196=421. Then remains 252 for 2 BCD. wherefore BCD will be 126. divide this by BC 14, then will 9 be found out for DC. From whence it follows AD = $\sqrt{}$: 225 − 81 = 12.

PROP. XIV.



To find a square ML equal to a right-lined figure given A.

a 45. 1. *a* Make the rectangle DB = A, and produce the greater side thereof DC to F, so that CF = CB.

b 10. 1. *b* bisect DF in G, about which as the center at the distance of GF describe the circle FHD, and draw out CB till it touch the circumference in H. Then shall be CHq = * ML = A.

* 46. 1.

c *constr.* For let GH be drawn. Then is A *c* = DB *c*

d 5. 2. *and* 3. *ax.* = DCF *d* = GFq − GCq *e* = HCq *c* = ML. *Which was to be done.*

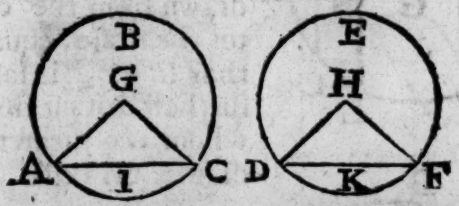
e 47. 1. *and*

The End of the second Book.

THE

THE THIRD BOOK OF EUCLIDE's ELEMENTS.

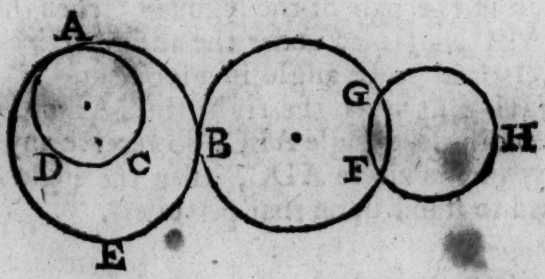
Definitions.



I. EQual circles (GABC, HDEF) are ſuch whoſe diameters are equal; or, from whoſe centers right lines drawn GA, HD, are equal.

II. A right line AB is ſaid to touch a circle FED, when touching the ſame, and being produced, it cutteth it not.

The right line FG cuts the circle FED.

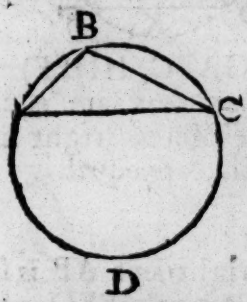


III. Circles DAC, ABE (and alſo FBG, ABE) are

are ſaid to touch one to the other, which touch, but cut not one the other.

The circle BFG cuts the circle FCH

IV. In a circle GABD, right lines FE, KL are ſaid to be equally diſtant from the center, when perpendiculars G H, G N drawn from the center G to them are equal. And that line BC is ſaid to be furtheſt diſtant from it, on whom the greater perpendicular GI falls.



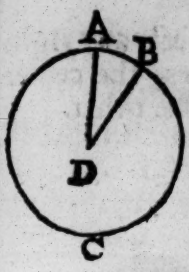
V. A ſegment of a circle (ABC) is a figure contained under a right line AC, and a portion of the circumference of a circle ABC.

VI. An angle of a ſegment CAB, is that angle which is contained under a right line CA and an arch of a circle AB.

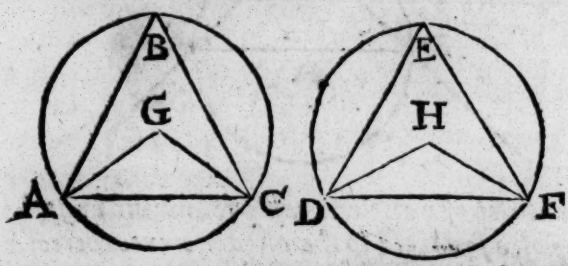
VII. An angle ABC is ſaid to be in a ſegment ABC, when in the circumference thereof ſome point B is taken, and from it right lines AB, CB, drawn to the ends of the right line AC, which is the baſe of the ſegment; then the angle ABC contained under the adjoined lines AB, CB, is ſaid to be angle in a ſegment.

VIII. But when the right lines AB, BC comprehending the angle ABC, do receive any periphery of the circle ADC, then the angle ABC is ſaid to ſtand upon that periphery.

IX.

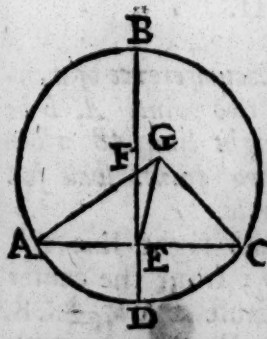


IX. A ſector of a circle (ADB) is when an angle ADB is ſet at the center D of that circle; namely, that figure ADB, comprehended under the right lines AD, BD containing the angle, and the part of the circumference received by them AB.



X. Like ſegments of a circle (ABC, DEF) are thoſe which include equal angles (ABC, DEF;) or, in whom the angles ABC, DEF are equal.

PROP. I.



To find the center F of a circle given ABC.

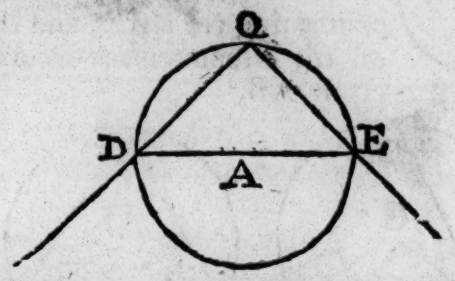
Draw a right line A C any-wiſe in the circle, which biſect in E, thro' E draw a perpendicular DB, and biſect the ſame in F, the point F ſhall be the center.

If you deny it, let G a point without the line DB be the center, (for it cannot be in the line BD, ſince that cannot be divided equally in any point but F;) let the lines GA, GC GE drawn. Now if G be the center, *a* then is GA=GC, and AE=EC by conſtruction, and the ſide GE common. *b* Therefore are the angles GEA, GEC equal, and *c* conſequently right. *d* Therefore the angle GEC=FEC: *e Which is abſurd.*

a 15. *def.* 1.
b 8. 1.
c 10. *def.* 1.
d 12. *ax.*
e 9. *ax.*

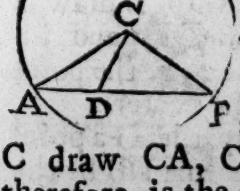
Coroll.

Hence, if a right line BD biſect any right line AC in a circle at right angles, the center ſhall be in the line BD that cuts the other.



Andr. Tacq. *The center of a circle is eaſily found out by applying the top of a ſquare Q to the circumference thereof.* For if the right line DE that joins the points D, E, in which the ſides of the ſquare QD, QE cut the circumference, be biſected in A, the point A ſhall be the center. The demonſtration whereof depends upon Prop. 31. of this Book.

PROP. II.



If in the circumference of a circle CAB any two points A, B be taken, the right line AB which joins thoſe two points ſhall fall within the circle.

Take in the right line AB any point D; from the center C draw CA, CD, CB. Becauſe CA *a* = CB, therefore is the angle A *b* = B. But the angle CDB *c* ⊏ A, therefore is CDB ⊏ B, therefore CB *d* ⊏ CD. But CB only reaches the circumference, therefore CD comes not ſo far; wherefore the point D is within the circle. The ſame may be proved of any other point in the line AB. And therefore the whole line AB falls within the circle. *Which was to be dem.*

a 15. *def.* 1.
b 5. 1.
c 16. 1.
d 19. 1.

Coroll.

Coroll.

Hence, if a right line touch a circle, ſo that it cut it not, it touches but in one point.

PROP. III.

If in a circle EABC a right line BD drawn thro' the center, biſect any other line AC not drawn through the center, it ſhall alſo cut it at right angles in F: And if it cuts it at right angles, it ſhall alſo biſect the ſame.

From the center E let the lines E A, E C be drawn.

1. *Hyp.* Becauſe A F *a* = F C, and E A *b* = E C, and the ſide EF common; the angles EFA, EFC *c* ſhall be equal, and *d* conſequently right. *Which was to be dem.*

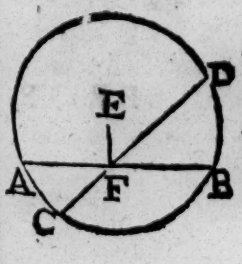
Hyp. 2. Becauſe EFA *e* = EFC, and the angle EAF *f* = ECF, and the ſide EF common; *g* therefore is AF = FC. Therefore is AC cut into two equal parts. *Which was to be dem.*

a *hyp.*
b 15. *def.* 1.
c 8. 1.
d 10. *def.* 1.
e *hyp. and* 12. *ax.*
f 5. 1.
g 26. 1.

Coroll.

Hence, in any equilateral or Iſoſceles triangle, if a line drawn from the vertical angle biſect the baſe, that line is perpendicular to it. And on the contrary, a perpendicular drawn from the vertical angle biſects the baſe.

PROP. IV.



If in a circle ACD two right lines AB, CD cut thro' one another, yet neither of them paſs thro' the center E, then neither of thoſe lines are divided into equal parts.

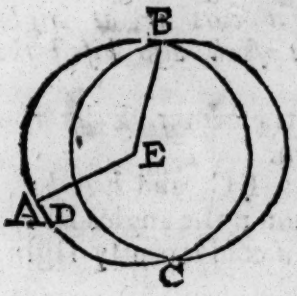
For if one line paſs thro'

the center, it appears that it cannot be biſected by the other; becauſe by Hypotheſis, the other does not paſs thro' the center.

If neither of them paſs thro' the center, then from the center E draw EF: now if AB, CD were both biſected in F, then *a* would the angles EFB, EFD be both right, and conſequently equal. *b Which is abſurd.*

a 3. 3.
b 9. *ax.*

PROP. V.

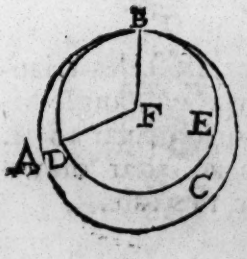


If two circles B A C, B D C cut one the other, they ſhall not have the ſame center E.

For otherwiſe the lines E B, EDA drawn from E the common center, would D E be *a* = EB *a* = E A. *b Which is abſurd.*

a 15. *def.* 1.
b 9. *ax.*

PROP. VI.



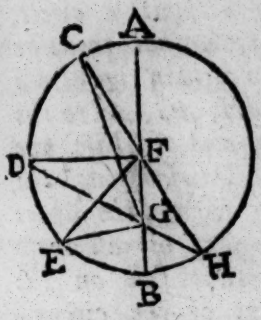
If two circles B AC, B D E, inwardly touch one the other (in B) *they have not one and the ſame center F.*

For otherwiſe the right lines FB, FDA drawn from the center F, would be FD *a* = FB *a* = FA, *b Which is abſurd.*

a 15. *def.* 1.
b 9. *ax.*

PROP.

PROP. VII.



If in A B the diameter of a circle, ſome point G be taken, which is not the center of the circle, and from that point certain right lines GC, G D, G E fall on the circle, the greateſt line ſhall be that (GA) in which is the center F; the leaſt, the remainder of the ſame line (GB) And of all the other lines, the line G C neareſt to that which was drawn thro' the center is always greater than any line farther removed G D; and only two lines are equal GE, GH, which fall upon the circle from the ſame point, on each ſide of the leaſt GB or of the greateſt GA.

From the center F draw the right lines FC, FD, FE; and *a* make the angle BFH=BFE. a 23. 1.

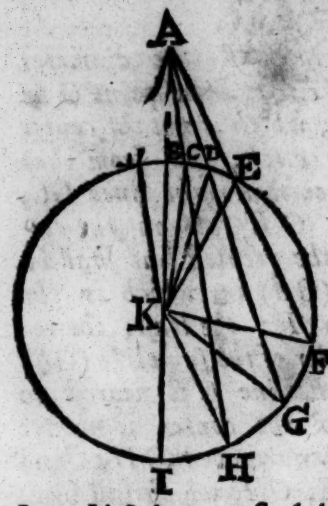
1. GF + FC (that is GA) *a* ⊏ GC. *Which* a 20. 1. *was to be dem.*

2. The ſide FG is common, and FC *b* = FD, b 15. *def.* 1. and the angle GFC *c* ⊏ GFD; *d* wherefore the c 9. *ax.* baſe GC ⊏ GD. d 24. 1.

3. FB (FE) *e* ⊐ GE + GF. *f* Therefore FG, e 20. which is common, being taken away from both, f 5. *ax.* BG ⊐ EG.

4. The ſide FG is common, and FE=FH, and the angle BFH *g* = BFE; *h* Therefore is GE = g *conſtr* GH. But that no other line GD from the point h 4. 1. G can be equal to GE, or GH, is already proved. *Which was to be dem.*

PROP. VIII.



If some point A be taken without a circle, and from that point be drawn certain right lines AI, AH, AG, AF to the circle, and of those one AI be drawn through the center K, and the others any-wise; of all those lines that fall on the concave of the circumference, that is the greatest AI which is drawn through the center; and of the others, that which is nearest (AH) to the line that passes through the center is greater than that which is more distant AG. But of all those lines that fall on the convex part of the circle, the least is that AB which is drawn from the point A to the diameter IB; and of the others, that (AC) which is nearest to the least, is less than that which is farther distant AD. And from that point there can be only two equal right lines AC, AL drawn, which shall fall on the circumference on each side of the least line AB or of the greatest AI.

From the center K draw the right lines KH, KG, KF, KC, KD, KE. and make the angle AKL = AKC.

a 20. 1. 1. AI (AK + KH) *a* ⊏ AH.

2. The side AK is common, and KH = KG,

b 24. 1. and the angle AKH ⊏ AKG; *b* therefore the base AH ⊏ AG.

c 20. 1. 3. KA *c* ⊐ KC + CA. From hence take away KC, KB that are equal; then will remain

d 5. *ax.* AB *d* ⊐ AC.

e 21. 1. 4. AC + CK *e* ⊐ AD + DK. From thence take away CK, DK that are equal; then re-

f 5. *ax.* mains AC *f* ⊐ AD.

5. The

5. The ſide KA is common, and KL = KC, and the angle AKL *g* = AKC; *h* therefore LA = *a* CA. But that no other line could be drawn equal to theſe, was proved above. Therefore, *&c.*

g *conſtr.*
h 4. 1.

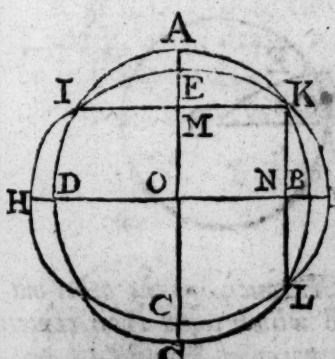
PROP. IX.

If in a circle BCK a point A be taken, and from that point more than two equal right lines AB, AC, AK, drawn to the circumference, then is that point A the center of the circle.

For *a* from no point without the center can more than two right lines equal be drawn to the circumference. Therefore A is the center. *Which was to be dem.*

a 7. 3.

PROP. X.



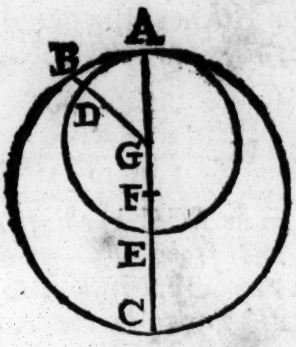
A circle I A K B L cannot cut another circle I E K F L in more than two points.

Let one circle, if it may be, cut the other in three points, I, K, L, and I K, K L being join'd, let them be biſected in M and N. *a* Both circles have their centers in their perpendiculars MC, NH, and in the interſection of thoſe perpendiculars, which is O. *b* Therefore the circles that cut each other have the ſame center. Which is falſe, by Prop. 5. 3.

a *cor.* 1. 3.
b 5. 3.

PROP.

PROP. XI.

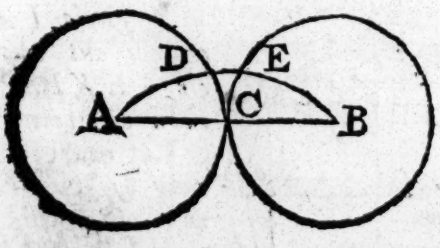


If two circles GADE, FABC touch one the other inwardly, and their centers be taken G, F; a right line FG joining their centers, and produced, ſhall cut the circumference in A the point of contact of the circles.

If it can be, let the right line FG produced cut the circles in ſome other point than A. ſo that not FGA, but FGDB ſhall be a right line. Let the line GA be drawn. Now, becauſe GD *a* =GA, and GB *b* ⌐ GA (ſince the right line FGB paſſes through F the center of the greater circle) therefore is GB ⌐ GD. *c Which is abſurd.*

a 15. *def.* 1.
b 7. 3.
c 9. *ax.*

PROP. XII.



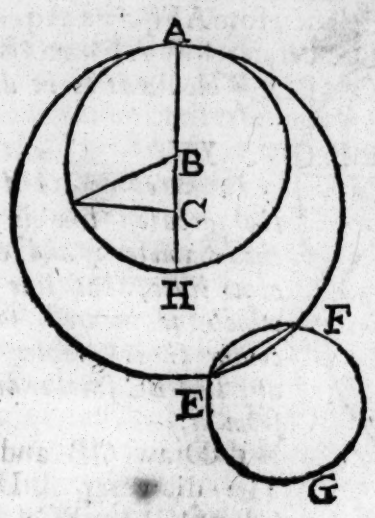
If two circles ACD, BCE touch one the other outwardly, the right line AB which joins their centers A, B, ſhall paſs thro' the point of contact C.

If it may be, let ADEB be a right line cutting the circles not in the point of contact C, but in the points D, E; draw AC, CB. then is AD + EB (AC + CB) *a* ⌐ ADEB. *b Which is abſurd.*

a 20. 1.
b 9. *ax.*

PROP.

PROP. XIII.



A circle C A F cannot touch a circle B A H in more points than one A, whether it be inwardly or outwardly.

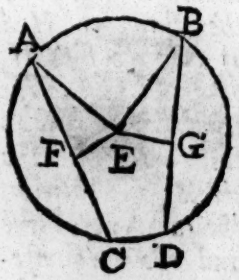
1. Let one circle (if it can be) touch another in two points A, H, *a* Then will the right line C B that joins the centers, if produced, fall as well in A, as H. Now becauſe CH *b*=CA, and BH ⊏ CH, therefore is BA (*c* BH) ⊏ CA. *d Which is abſurd.*

a 11. 3.
b 15. *def.* 1.
c 15. *def.* 1.
d 9. *ax.*

2. If it be ſaid to touch outwardly in the points E and F, then draw the line EF, *e* which will be in both circles, Therefore thoſe circles cut one the other; Which is againſt the Hyp.

e 2. 3.

PROP. XIV.



In a circle E A B C equal right lines A C, B D are equally diſtant from the center E: and right lines A C, B D which are equally diſtant from the center, are equal among themſelves.

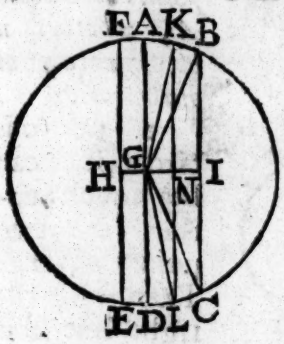
From the center E draw the perpendiculars EF, EG. *a* which will biſect the lines AC, BD. join EA, EB. 1. *Hyp.* AC=BD. therefore AF *b*=BG. But al-

a 3. 3.
b 7. *ax.*

ſo

ſo EA = EB. therefore EFq *c* = EAq − AFq =
c 47.1. & EBq − BGq *c* = EGq. *d* Therefore FE = EG.
3. *ax.* 2. *Hyp.* EF = EG. Therefore AFq *c* = EAq − EFq
d *ſch.* 48.1. = EBq − EGq = BGq. Therefore AF *d* = GB, and
e 6. *ax.* *e* conſequently AC = BD. *Which was to be dem.*

PROP. XV.

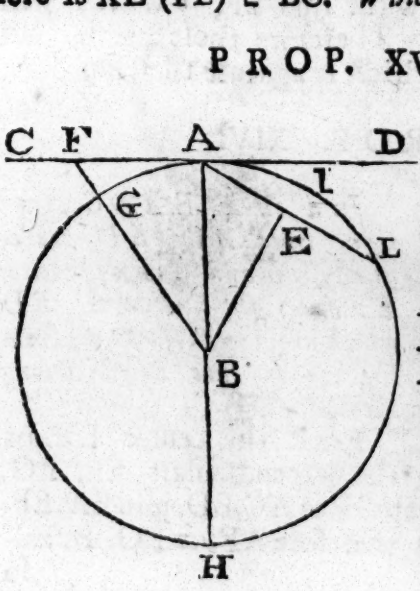


In a circle G A B C the greateſt line is A D the diameter; and of all other lines, that line FE, which is neareſt to the center G is greater than any line BC farther diſtant from it.

a 15. *def.* 1. b 20. 1. 1. Draw GB and GC. The diameter A D (*a* GB + GC) *b* ⊏ BC.

2. Let the diſtance GI be ⊏ GH. Take GN = GH. Thro' the point N draw KL perpendicularly to GI: Join GK, GL. Becauſe GK = GB, and GL
c 24. 1. = GC, and the angle KGL ⊏ BGC; *c* therefore is KL (FE) ⊏ BC. *Which was to be dem.*

PROP. XVI.



A line C D drawn from the extreme point of the diameter H A of a circle B A L H, perpendicular to the ſaid diameter, ſhall fall without the circle; and between the ſame right line and the circumference cannot be drawn another line A L. And the angle of

of the ſemicircle BAI, is greater than any right-lined acute angle BAL; and the remaining angle without the circumference DAI is leſs than any right-lined angle.

1. From the center B to any point F in the right line AC, draw the right line BF. The ſide BF ſubtending the right angle BAF is *a* greater than the ſide BA which is oppoſite to the acute angle BFA. Therefore whereas BA (BG) reaches to the circumference, BF ſhall reach further; and ſo the point F. and for the ſame reaſon is any other point of the line AC placed without the circle. a 19. 1.

2. Draw BE perpendicular to AL. The ſide BA oppoſite to the right angle BEA is *b* greater than the ſide BE which ſubtends the acute angle BAE; therefore the point E, and ſo the whole line EA falls within the circle. b 19. 1.

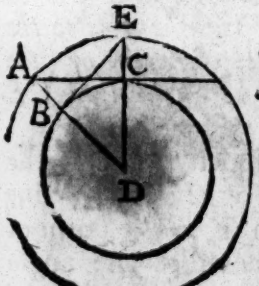
3. Hence it follows that any acute angle, to wit, EAD, is greater than the angle of contact DAI. and that any acute angle BAL is leſs than the angle of a ſemicircle BAI. *Which was to be dem.*

Coroll.

Hence, A right line drawn from the extremity of the diameter of a circle, and at right angles, is a tangent to the ſaid circle.

From this Propoſition are gathered many Paradox and wonderful Conſectaries, which you may meet with in the Interpreters.

PROP. XVII.



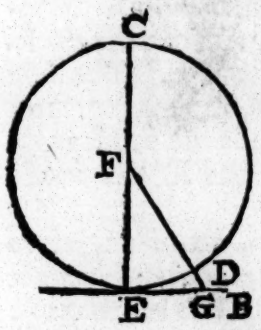
From a point given A to draw a right line AC which ſhall touch a circle given DBC.

From D the center of the circle given let a line DA cutting the circumference in B, be drawn to the point given A; from the center D deſcribe another circle thro' the point A; and from B draw a perpendicular to AD, which ſhall meet with the circle AE in the

the point E; and draw ED meeting with the circle BC in the point C. Then the line drawn from A to C ſhall touch the circle DBC.

a 15. *def.* 1. b 4. 1. c *cor.* 16. 3. For DB *a* = DC, and DE *a* = DA, and the angle D is common; *b* therefore the angle ACD EBD and right. *c* Therefore AC touches the circle in C. *Which was to be done.*

PROP. XVIII.



If any right line AB touch a circle FEDC, and from the center to the point of contact E a right line FE be drawn; that line FE ſhall be perpendicular to the tangent AB.

If you deny it, let ſome other line FG be drawn from the center F perpendicular to the tangent, and *a* cutting the circle in D. Therefore, whereas the angle FGE is ſaid to be right, *b* thence is the angle FEG acute; *c* ſo that FE (FD) ⊏ FG. *Which is abſurd.*

a 2. *def.* 3. & 16 3. b *cor.* 17. 1. c 19. 1. d 9. *ax.*

PROP. XIX.

If any right line AB touch a circle, and from the point of contact C a right line CE be erected at right angles to the tangent, the center of the circle ſhall be in the line CE ſo erected.

If you deny it, let the center be without the line CE in the point F; and from F to the point of contact let FC be drawn. Therefore the angle *FCB is right, and *a* conſequently equal to the angle ECB, which was right by Hyp. *b Which is abſurd.*

* 18. 3. a 11. *ax.* b 9. *ax.*

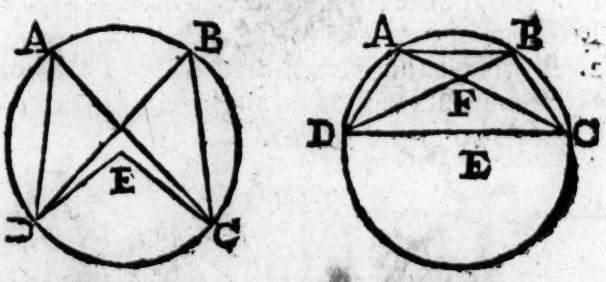
PROP.

PROP. XX.

In a circle DABC, the angle BDC at the center is double of the angle BAC at the circumference, when the ſame arch of the circle BC is the baſe of the angles.

Draw the diameter ADE. The outward angle BDE *a* = DAB + DBA *b* = 2 DAB. Likewiſe a 32. 1. the angle EDC = 2 DAC. Therefore in the firſt b 5. 1. caſe the whole angle *c* BDC = 2 BAC. and in c 2. *ax.* the third caſe the *d* remaining angle BDC = 2 d 20. *ax.* BAC. *Which was to be dem.*

PROP. XXI.



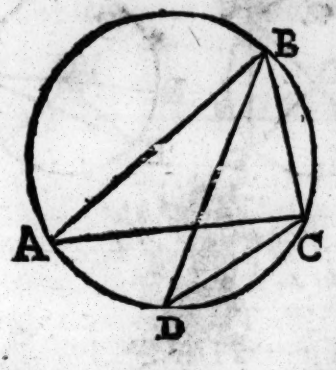
In a circle EDAC, the angles DAC and DBC which are in the ſame ſegment, are equal one to the other.

1. *Caſe.* If the ſegment DABC be greater than a ſemicircle, from the center E draw ED, EC. Then is twice the angle A *a* = E *a* = 2 B. *W. W. to be dem.* a 20. 3.

2. *Caſe.* If the ſegment be leſs than a ſemicircle, then is the ſum of the angles of the triangle ADF equal to the ſum of the angles of the triangle BCF. From each let AFD be taken away *b* equal to b 15. 1. BFC,

c *By the case.* 1. BFC, and ADB c = ACB be likewiſe taken away, then remains DAC = DBC. *W. W. to be dem.*

PROP. XXII.



The angles A D C, ABC of a quadrilateral figure ABCD deſcribed in a circle, which are oppoſite one to the other, are equal to two right angles.

Draw A C, B D. The angle A B C + B C A + BAC a = 2 right. But B D A b = BCA, and BDC b = BAC. c Therefore ABC + ADC = 2 right angles. *Which was to be dem.*

a 32. 1.
b 21. 3.
c 1. *ax.*

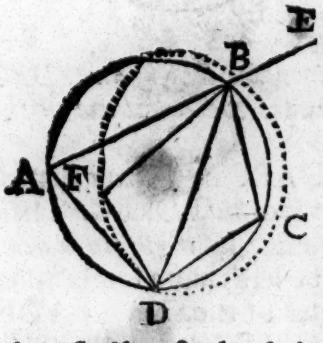
Coroll.

1. Hence, If one ſide * AB of a quadrilateral deſcribed in a circle be produced, the external angle EBC is equal to the internal angle ADC, which is oppoſite to that ABC which is adjacent to EBC, as appears by 13. 1. and 3. *ax.*

* *See the following Diagr.*

2. A circle cannot be deſcribed about a Rhombus; becauſe its oppoſite angles are greater or leſs than two right angles.

Schol.



If in a quadrilateral ABCD the angles A and C, which are oppoſite, be equal to two right, then a circle may be deſcribed about that quadrilateral.

For a circle will paſs thro' any three angles (as ſhall appear by 5. 4.) I ſay that ſhall paſs thro' A the fourth alſo of ſuch

a quadrilateral: For if you deny it, let the circle paſs thro' F. Therefore the right lines BF, FD, BD being drawn, the angle C + F *a* = 2 right *b* = C + A wherefore, A *c* is equal to F. *d Which is abſurd.*

a 22. 3.
b *hyp.*
c 3. *ax.*
d 21. 1.

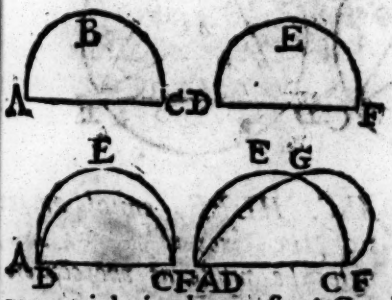
PROP. XXIII.

Two like and unequal ſegments of circles ABC, ADC cannot be ſet on the ſame right line AC, and the ſame ſide thereof.

For if they are ſaid to be like, draw the line CB cutting the circumference in D and B, join AB and AD. Becauſe the ſegments are ſuppoſed like, *a* therefore is the angle ADC = ABC. *b Which is abſurd.*

a 10. *def.* 3.
b 16. 1.

PROP. XXIV.

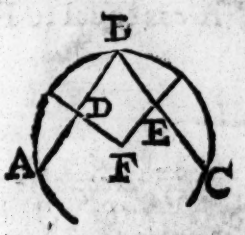


Like ſegments of circles A B C, D E F upon equal right lines A C, D F, are equal one to the other.

The baſe AC being laid on the baſe DF will agree with it, becauſe AC = DF. Therefore the ſegment ABC ſhall agree with the ſegment DEF (for otherwiſe it ſhall fall either within or without, and if ſo *a* then the ſegments are not like, which is contrary to the Hypotheſis, and at leaſt it ſhall fall partly within and partly without, and ſo cut in three points, *b* which is abſurd. *c* Therefore the ſegments ABC = DEF. *Which was to be dem.*

a 23. 3.
b 10. 3.
c 8. *ax.*

PROP. XXV.

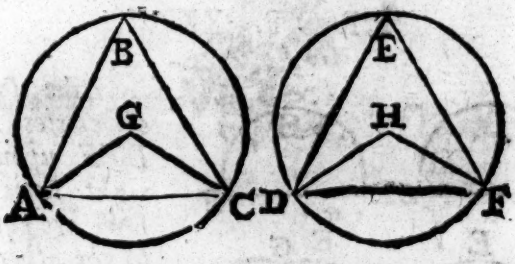


A ſegment of a circle ABC being given, to deſcribe the whole circle whereof that is a ſegment.

Let two right lines be drawn AB, BC, which biſect in the points D and E. From D and E draw the perpendiculars DF, EF meeting in the point F. I ſay this point ſhall be the center of the circle.

a *cor.* 1. 3. For the center ſhall be as well in *a* DF as; FE, therefore it muſt be in the common point F. *Which was to be done.*

PROP. XXVI.

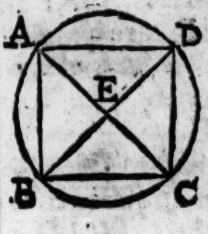


In equal circles GABC, HDEF, equal angles ſtand upon equal parts of the circumference, AC, DF; whether thoſe angles be made at the centers, G, H, or at the circumferences, B, E.

Becauſe the circles are equal, therefore is GA = HD, and GC = HF; alſo by Hypotheſis
a *def.* 1. the angle G = H; *a* therefore AC = DF.
b 20. 3. Moreover the angle B *b* = ½ G *c* = ½ H *b* = E.
c *hyp.* *d* Therefore the ſegments ABC, DEF are like,
d 10. *def.* 3. and *e* conſequently equal, *f* whence the remain-
e 24. 3. ing ſegments alſo AC, DF are equal. *Which*
f 3. *ax.* *was to be dem.*

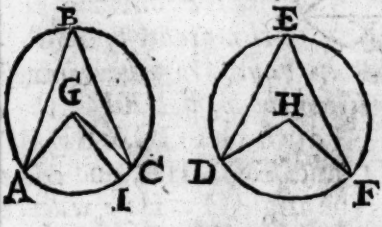
Coroll.

Coroll.



In a circle ABCD let an arch AB be equal to DC; then ſhall AD be parallel to BC. For the right line AC being drawn, the angle ACB *a* = CAD; where- a 26. 3.
fore by 27. 1. the ſaid ſides are parallel.

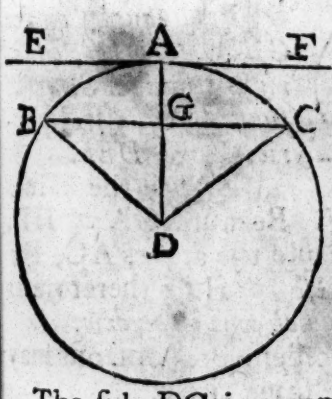
PROP. XXVII.



In equal circles GABC, HDEF, the angles ſtanding upon equal parts of the circumference AC, DF, are equal between themſelves, whether they be made at the centers G, H, or at the circumferences, B, E.

For if it be poſſible, let one of the angles AGC be ⊏ DHF, and make AGI = DHF. thence is the arch AI *a* = DF *b* = AC. *Which is abſurd.* a 26. 3.
b *hyp.*
c 9. *ax.*

Schol.



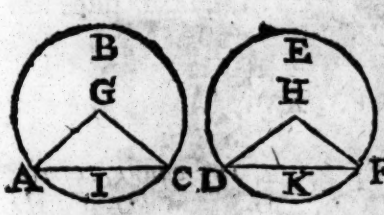
A right line EF, which being drawn from A the middle point of any periphery BC, toucheth the circle, is parallel to the right line BC ſubtending the ſaid periphery.

From the center D draw a right line DA to the point of contact A, and join DB, DC.

The ſide DG is common, and DB = DC, and the angle BDA *a* = CDA, becauſe the arches BA, a 27. 3.

b *hyp.* CA are *b* equal) therefore the angles at the baſe
c 4. 1. DGB, DGC are *c* equal, and *d* conſequently
d 10. *def.* 1. right; But the inward angles GAE, GAF are
e *hyp.* alſo *e* right, *f* therefore BC, EF are parallel.
f 28. 1. *Which was to be dem.*

PROP. XXVIII.



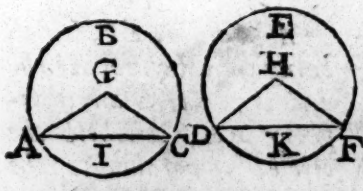
In equal circles GABC, HDEF, equal right lines AC, DF cut off equal parts of the circumference, the greateſt ABC equal to the greateſt DEF, and the leaſt AIC to the leaſt DKF.

From the centers G, H draw GA GC, & HD, HF

a *hyp.* Becauſe GA = HD, and GC = HF, and AC *a*
b 8. 1. = DF, *b* therefore is the angle G = H; *c* whence
c 26. 3. the arch AIC = DKF; *d* and ſo the remaining
d 3. *ax.* arch ABC = DEF. *Which was to be dem.*

But if the ſubtended line AC be ⊏ or ¬ than DF, then in like manner will the arch AC be ⊏ or ¬ than DF.

PROP. XXIX.



In equal circles GABC, HDEF, equal right lines AC, DF ſubtend equal peripheries ABC, DEF.

Draw the lines GA, GC, and HD, HF. Becauſe GA = HD, and GC = HF, and (becauſe the arches AC, DF

a *hyp.* are *a* equal) the angle G *b* = H. *c* therefore is
b 27. 3. the baſe AC = DF. *Which was to be dem.*
c 4. 1.

This and the three precedent Propoſitions may be underſtood alſo of the ſame circle.

PROP.

PROP. XXX.

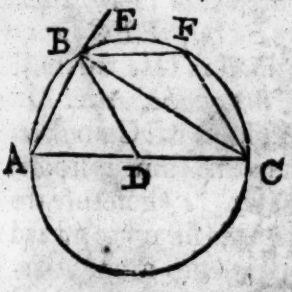
To cut a periphery given ABC into two equal parts.

Draw the right line AC, and biſect it in D; from D draw a perpendicular DB meeting with the arch in B, it ſhall biſect the ſame.

For join AB, and CB. The ſide DB is common, and AD *a* = DC, and the angle ADB *b* = CDB. *c* therefore AB = BC; *d* whence the arch AB = BC. *Which was to be done.*

a *conſtr.*
b 12. *ax.*
c 4. 1.
d 28. 3.

PROP. XXXI.



In a circle, the angle ABC, which is in the ſemicircle, is a right angle; but the angle, which is in the greater ſegment BAC is leſs than a right angle; and the angle which is in the leſſer ſegment BFC is greater than a right angle. Moreover, the angle of the greater ſegment is greater than a right angle, and the angle of the leſſer ſegment is leſs than a right angle.

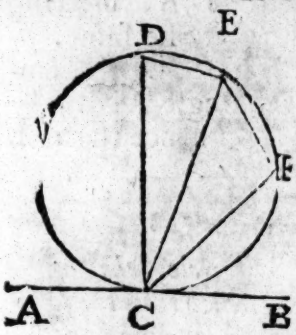
From the center D draw DB, becauſe DB = DA, therefore is the angle A *a* = DBA, and the angle DCB *a* = DBC, *b* therefore the angle ADC = A + ACB *c* = EBC, *d* ſo that ABC and EBC are right angles. *W. W. to be dem.* *e* Therefore BAC is an acute angle. *f W. W. to be dem.* And further, whereas BAC *f* + BFC = 2 right, therefore BFC is an obtuſe angle. Laſtly, the angle contained under the right line CB, and the arch BAC is greater than the right angle ABC; but the angle made by the right line CB and the periphery of the leſſer ſegment BFC *g* is leſs than the right angle ABC. *Which was to be dem.*

a 5. 1.
b 2. *ax.*
c 32. 1.
d 10. *def.* 1.
e *cor.* 17. 1.
f 22. 3.
g 9. *ax.*

Schol.

In a right-angled triangle ABC, if the Hypothenuſe (or ſubtended line) AC be biſected in D, a circle drawn from the center D thro' the point A ſhall alſo paſs thro' the point B: As you may eaſily demonſtrate from this Prop. and 21. 1.

PROP. XXXII.



If a right line AB touch a circle, and from the point of contact be drawn a right line CE cutting the circle, the angles ECB, ECA which it makes with the tangent line are equal to thoſe angles EDC, EFC which are made in the alternate ſegments of the circle.

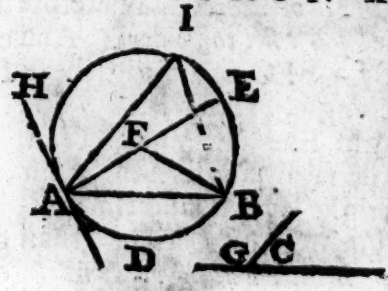
Let CD, the ſide of the angle EDC, be perpendicular to AB (*a* for it's to the ſame purpoſe) *b* therefore CD is the diameter. *c* therefore the angle CED in a ſemicircle is a right angle, *d* and therefore the angle D + DCE = to a right angle *e* = ECB + DCE. *f* Therefore the angle D = ECB. *Which was to be dem.*

a 26. 3.
b 19. 3.
c 31. 3.
d 32. 1.
e *conſtr.*
f 3. *ax.*

Now whereas the angle ECB + ECA *g* = 2 right *h* = D + F, from both of theſe take away ECB and D, which are equal, *k* then remains ECA = F. *Which was to be dem.*

g 13. 1.
h 22. 3.
k 3. *ax.*

PROP. XXXIII.



Upon a right line AB to deſcribe a ſegment of a circle AIEB which ſhall contain an angle AIB equal to a right lined angle given C.

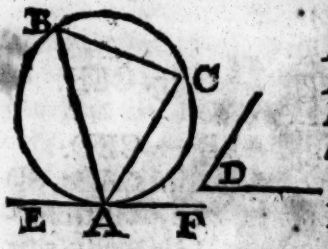
a Make the angle BAD

a 23. 1.

BAD = C. Thro' the point A draw the line AE perpendicular to HD. At one end of the line given AB make an angle ABF = BAF, one ſide whereof let it cut the line AE in F; from the center F thro' the point A, deſcribe a circle which ſhall paſs thro' B (becauſe the angles FBA *b* = FAB. and *c* therefore FB = FA,) AIB is the ſegment ſought. For becauſe HD is perpendicular to the diameter AE, it therefore touches the circle HD which AB cuts. And therefore the angle AIB *e* = BAD *f* = C. *Which was to be done.*

b *conſtr.* c 6. 1. d *cor.* 16. 3. e 32. 3. f *conſtr.*

PROP. XXXIV.

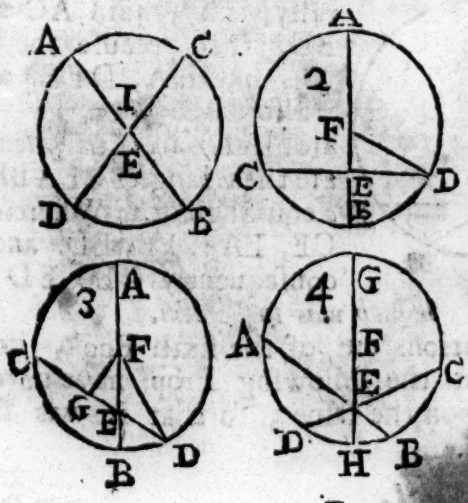


From a circle given ABC to cut off a ſegment ABC containing an angle B equal to a right-lined angle given D.

a Draw a right line EF which ſhall touch the circle given in A. *b* let AC be drawn alſo making an angle FAC = D. This line ſhall cut off ABC containing an angle B *c* = CAF *d* = D. *Which was to be done.*

a 17. 3. b 23. 1. c 32. 1. d *conſtr.*

PROP. XXXV.



If in a circle FB C A two right lines AB, DC cut each other, the rectangle comprehended under the ſegments AE, EB of the one, ſhall be equal to the

the rectangle comprehended under the ſegments CE, ED of the other.

1. *Caſe.* If the right lines cut one the other in the center, the thing is evident.

2. *Caſe.* If one line AB paſs thro' the center F, and biſect the other line CD, then draw FD.

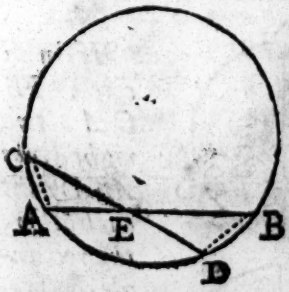
a 5. 2. Now the rectangle AEB + FEq *a* FBq *b* = FDq *c*
b *ſch.* 48. 1. = EDq + FEq *d* = CED + FEq. *e* Therefore the
c 47. 1. rectangle AEB = CED. *Which was to be dem.*
d *hyp.*

3. *Caſe.* If one of the lines AB be the diameter,
e 3. *ax.* and cut the other line CD unequally, biſect CD by FG a perpendicular from the center.

f 5. 2. These are equal, The rectangle AEB + FEq.
g 47. 1. *f* FBq (FDq.)
h 5. 2. *g* FGq + GDq.
k 47. 1. EGq + *b* GEq + Rectang. CED.
l 3. *ax.* *b* FEq + CED.

l Therefore the Rectangle AEB = CED.

4. *Caſe.* If neither of the right lines AB, CD paſs thro' the center, then thro' the point of interſection E, draw the diameter GH. By that which hath been already demonſtrated, it appears that the rectangle AEB = GEH = CED. *Which was to be dem.*

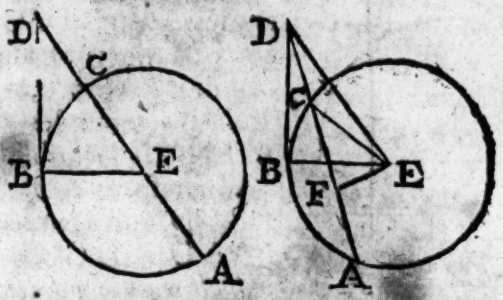


a 15. 1. More eaſily, and generally, thus; join AC and
b 21. 3. BD, then becauſe the angles *a* CEA, DEB, and
b alſo C, B (upon the ſame
c *cor.* 31. 1. arch AD) are equal, thence are the triangles CEA BED
d 4. 6. *c* equiangular. *d* Wherefore
e 16. 6. CE, EA :: EB, ED. and *e* conſequently CE × ED = AE × EB. *Which was to be dem.*

The citations out of the ſixth Book, both here and in the following Prop. have no dependance upon the ſame; ſo that it was free to uſe them.

PROP.

PROP. XXXVI.



If any point D be taken without a circle E B C, and from that point two right lines D A, D E fall upon the circle, whereof one D A cut the circle, the other D B touches it, the rectangle comprehended under the whole line D A that cuts the circle, and under D C that part which is taken from the point given D to the convex of the periphery, shall be equal to the square made of the tangent line.

1. *Case.* If the secant AD pass thro' the center, then join EB, this *a* will make a right angle with the line DB, wherefore DBq + *d* EBq (ECq) *b* = EDq *c* = AD × DC + ECq. Therefore AD × DC = DBq. *Which was to be dem.*

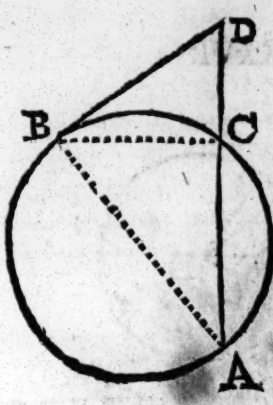
a 18. 1.
b 47. 1.
c 6. 2.
d 3. *ax.*

2. *Case.* But if AD pass not thro' the center then draw EC, EB, ED, and EF perpendicular to AD, *a* wherefore AC is bisected in F.

Because BDq + EBq *b* = DEq *b* = EFq + FDq *c* = EFq + ADC + FCq *d* = ADC + CEq (EBq.) *e* Therefore is BDq = ADC. *Which was to be dem.*

a 3. 3.
b 47. 1.
c 6. 2.
d 47. 1.
e 3. *ax.*

More

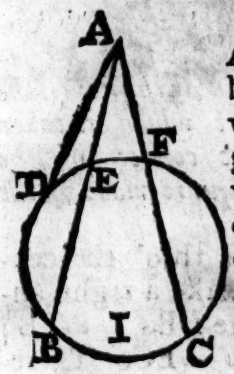


a 32. 3. b 32. 1. c 4. 6. d 17. 6.

More eaſily, and generally, thus; Draw AB and BC. Then becauſe the angles A, and DBC *a* are equal, and the angle D common to both, thence are the triangles BDC, ADB *b* equiangular, *c* Wherefore AD. DB :: DB.CD. and *d* conſequently AD × DC = DBq. *Which was to be dem.*

Coroll.

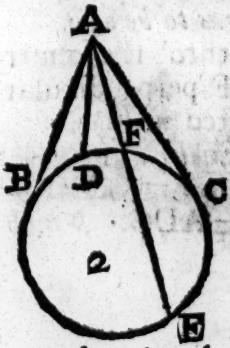
1. Hence, If from any point A taken without a circle, there be ſeveral lines AB, AC drawn which cut the circle, the rectangles comprehended under the whole lines AB, AC, and the outward parts AE, AF are equal between themſelves.



a 36. 3.

For if the tangent AD be drawn, then is CAF = ADq *a* = BAE.

2. It appears alſo from hence, that if two lines AB, AC drawn from the ſame point do touch a circle, thoſe two lines are equal one to the other.



a 36. 3. b 36. 3.

For if AE be drawn cutting the circle, then is ABq *a* = EAF *b* = ACq.

3. It is alſo evident that from a point A taken without a circle, there can be drawn but two lines AB, AC that ſhall touch the circle.

For if a third line AD be ſaid to touch the circle, thence is AD *c* = AB *c* = AC. *d Which is abſurd.*

c 2. *cor.* d 8. 3.

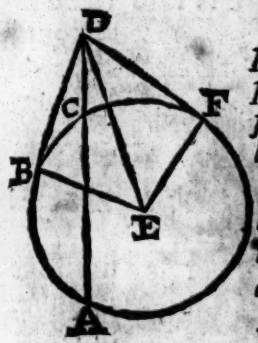
4. And

4. And on the contrary, it is plain, that if two equal right lines AB, AC fall from any point A upon the convex periphery of a circle, and that if one of these equal lines AB touch the circle, then the other AC touches the circle also.

For if possible, let not AC, but another line AD, touch the circle; therefore is AD *e* = AC *f* AB. *g Which is absurd.*

e 2. *cor.*
f *hyp.*
g 8. 3.

PROP. XXXVIII.



If without a circle EBF any point D be taken, and from that point two right lines DA, DB fall on the circle, whereof one line DA cuts the circle the other DB falls upon it; and if also the rectangle comprehended under the whole line that cuts the circle, and under that part of it DC which is taken betwixt the point D and the convex periphery, be equal to that square which is made of the line DB falling on the circle, I say that the line DB so falling shall touch the circle given.

From that point D *a* let a tangent DF be drawn, and from the center E draw ED, EB, EF. Now because DBq *b* = ADC *c* = DFq, therefore is DB *d* = DF: But EB = EF, and the side ED common; *e* therefore the angle EBD = EFD. but EFD is a right angle, and *f* therefore EBD is right also, and *g* therefore DB touches the circle. *Which was to be dem.*

a 17. 3.
b *hyp.*
c 36. 3.
d 1. *ax.* & *sch.* 4. 1.
e 8. 1.
g *cor.* 16. 3.
h 8. 1.

Coroll.

From hence it follows that the *h* angle EDB = EDF.

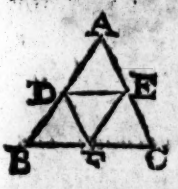
The End of the third Book.

THE

THE FOURTH BOOK OF EUCLIDE's ELEMENTS.

Definitions.

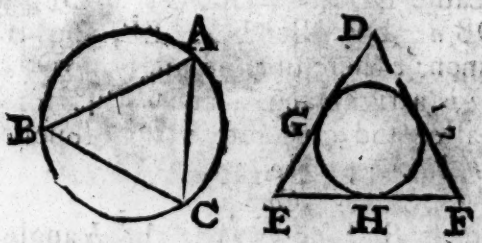
I. A Right-lined figure is ſaid to be inſcribed in a right-lined figure, when every one of the angles of the inſcribed figure touch every one of the ſides of the figure wherein it is inſcribed.



So the triangle DEF is inſcribed in the triangle ABC.

II. In like manner a figure is ſaid to be deſcribed about a figure, when every one of the ſides of the figure circumſcribed touch every one of the angles of the figure about which it is circumſcribed.

So the triangle ABC is deſcribed about the triangle DEF.



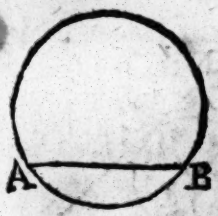
III. A right-lined figure is ſaid to be inſcribed in a circle, when all the angles of that figure which is inſcribed do touch the circumference of the circle.

IV. A right-lined figure is ſaid to be deſcribed about a circle, when all the ſides of the figure which

which is circumſcribed touch the periphery of the circle.

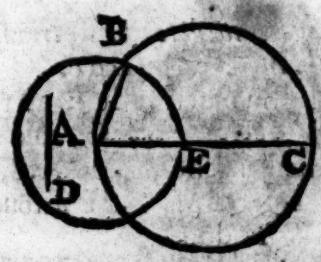
V. After the like manner a circle is ſaid to be inſcribed in a right-lined figure, when the periphery of the circle touches all the ſides of the figure in which it is inſcribed.

VI. A circle is ſaid to be deſcribed about a figure, when the periphery of the circle touches all the angles of the figure, which it circumſcribes.



VII. A right line is ſaid to be co-apted or applied in a circle when the extremes thereof fall upon the circumference; *as the right line AB.*

PROP. I. *Probl.* 1.

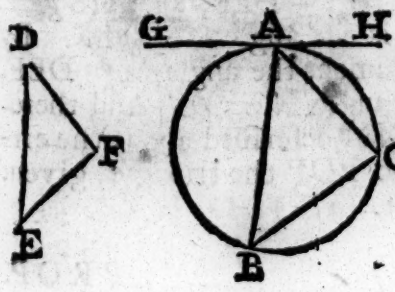


In a circle given ABC to apply a right line AE equal to a right line given D, which doth not exceed AC the diameter of the circle.

From the center A by the ſpace AE = D *a* deſcribe a circle meeting with the circle given in B, draw AB. Then is AB *b* = AE *c* = D. *Which was to be done.*

a 3. *poſt.* & 3. 1.
b 15. *def.* 1.
c *conſtr.*

PROP. II. *Probl.* 2.



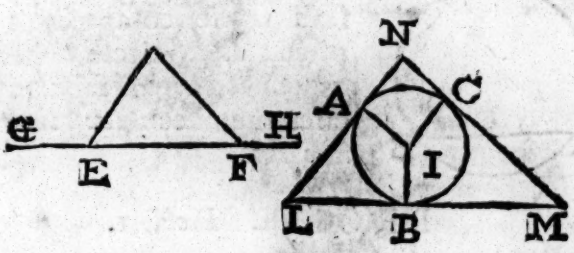
In a circle given ABC to deſcribe a triangle ABC, equiangular to a triangle given D. EF.

Let the right line

a 17. 3. b 23. 1. line GH *a* touch the circle given in A; *b* make the angle HAC = E, *b* and the angle GAB = F. then join BC; and the thing is done.

c 32. 3. d *constr.* e 32. 1. For the angle B *c* = HAC *d* = E, and the angle C *c* = GAB *d* = F; *e* whence alſo the angle BAC = D. Therefore the triangle BAC inſcribed in the circle is equiangular to DEF. *Which was to be done.*

PROP. III.



About a circle given IABC to deſcribe a triangle LNM equiangular to a triangle given DEF.

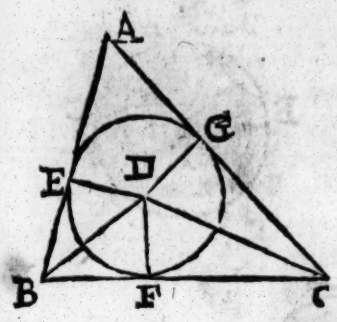
a 23. 1. b 17. 3. Produce the ſide EF on both ſides; at the center I *a* make an angle AIB = DEG, and an angle BIC = DFH. Then in the points A, B, C let three right lines LN, LM, NM *b* touch the circle, and the thing is done.

c 13. 1. d 11. 3. For it's evident that the right lines LN, LM, MN will meet and make a triangle, *c* becauſe the angles LAI, LBI are right; ſo that the *d* right line AB produced will make the angles LAB, LBA, leſs than two right angles.

a *ſch.* 32. 1. f 13. 1. g *conſtr.* h 3. *ax.* k 32. 1. Since, therefore the angle AIB + L *e* = 2 right angles *f* = DEG + DEF, and AIB *g* = DEG; *h* therefore is the angle L = DEF. By the like way of argument the angle M = DEF. *k* therefore alſo the angle N = D. And therefore the triangle LNM deſcribed about the circle is equiangular to EDF the triangle given. *Which was to be done.*

PROP.

PROP. IV.



In a triangle given ABC, to deſcribe a circle EFG.

a Biſect the angles B and C with the right lines BD, CD meeting in the point D, *b* and draw the perpendiculars D E, DF, DG. A circle deſcribed from the center D thro' E, will paſs thro' G and F, and touch the three ſides of the triangle. a 9. 1. b 12. 1.

For the angle BDE *c* = DBF; and the angle DEB *d* = DFB; and the ſide DB common. *e* therefore DE = DF. By the like argument DG = DF. The circle therefore deſcribed from the center D paſſes thro' the three points E, F, G. and whereas the angles at E, F, G are right, therefore it touches all the ſides of the triangle. *Which was to be done.* c *conſtr.* d 12. *ax.* e 26. 1.

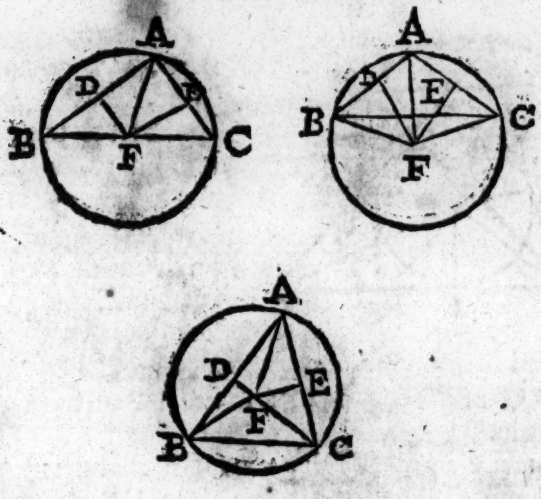
Schol.

Hence, *The ſides of a triangle being known, their ſegments which are made by the touchings of the circle inſcribed ſhall be found,* thus; Pet. Herig.

Let AB be 12, AC 18, BC 16, then is AB + BC = 28. Out of which ſubduct 18 = AC = AE + FC, then remains 10 = BE + BF. Therefore BE, or BF = 5; and conſequently FC, or CG, = 11. Wherefore GA, or AE, = 7.

PROP.

PROP. V.



About a triangle given ABC to deſcribe a circle FABC.

a 10. *and* 11. 1. *a* Biſect any two ſides BA, CA with a perpendicular DF, EF meeting in the point F. I ſay this ſhall be the center of the circle.

For, let the right lines FA, FB, FC be drawn. Now becauſe AD *b* = DB and the ſide DF common, and the angles FDA *c* = FDB, therefore is FB *d* = FA. After the ſame manner is FC = FA. Therefore a circle deſcribed from the center F ſhall paſs thro' the angles of the triangle given (*viz.*) B, A, C. *Which was to be done.*

b *conſtr.* c *conſtr.* & 11. *ax.* d 4. 1.

Coroll.

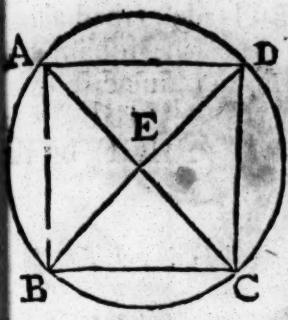
* 31. 3. * Hence, if a triangle be acute-angled, the center ſhall fall within the triangle; if right-angled, in the ſide oppoſite to the right angle, and if obtuſe-angled, without the triangle.

Schol.

By the ſame method may a circle be deſcribed, that ſhall paſs thro' three points given, not being in the ſame ſtrait line.

PROP.

PROP. VI.



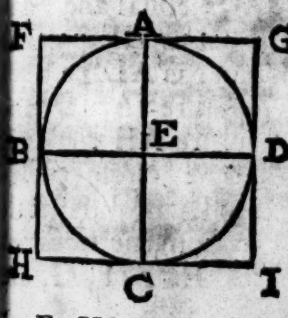
In a circle given EABCD to inſcribe a ſquare ABCD.

a Draw the diameters AC, BD cutting each other at right angles in the center E. Join the extremes of theſe diameters with the right lines AB, BC, CD, DA. *And the thing is done.* a 11. 1.

Now becauſe the four angles at E are right, the *b* arches and *c* ſubtended lines AB, BC, CD, DA are equal; therefore is the figure ABCD equilateral, and all the angles in ſemicircles, and ſo *d* right. *e* Therefore ABCD is a ſquare inſcribed in a circle given. *Which was to be done.*

b 26. 3.
c 29. 3.
d 31. 3.
e 29. 3. *def.* 1.

PROP. VII.



About a circle given EABCD to deſcribe a ſquare FHIG.

Draw the diameters AC, BD cutting one the other at right angles; thro' the extremes of theſe diameters *a* draw tangents meeting in F, H, I, G, then I ſay it's done. a 17. 3.

For becauſe *b* the angles A and C are right, *c* therefore is FG parallel to HI. After the ſame manner is FH parallel to GI. and therefore FHIG is a Pgr. and alſo right-angled. It is equilateral becauſe FG *d* = HI *d* - DB *e* CA *d* FH *d* = GI. Wherefore FHIG is a *f* ſquare deſcribed about the circle given. *Which was to be done.*

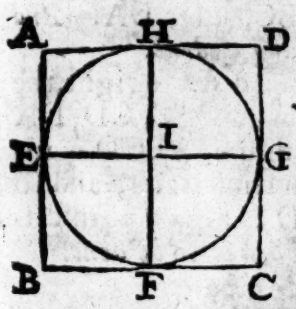
b 18. 3.
c 18. 1.
d 34. 1.
e 15. *def.* 1.
f 29. *def.* 1.

Schol.

A ſquare ABCD deſcribed about a circle is double of the ſquare EFGH inſcribed in the ſame circle.

For the rectangle HB = 2 DHEF, and HD = 2 HGF by the 41. 1.

PROP. VIII.



In a ſquare given ABCD to inſcribe a circle IEFGH.

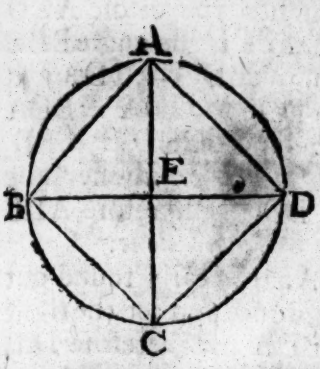
Biſect the ſides of the ſquare in the points H, E, F, G, cutting one the other in I. A circle drawn from the center I through H ſhall be inſcribed in the ſquare.

a 7. *ax.*
b *hyp.*
c 33. 1.
d 7. *ax.*
e 34. 1.

For becauſe AH and BF are *a* equal and *b* parallel, *c* therefore is AB parallel to HF parallel to DC. After the ſame manner is AD parallel to EG, parallel to BC; therefore IA, ID, IB, IC are parallelograms. Therefore AH *d* = AE *e* = HI = EI = FI = IG. The circle therefore deſcribed from the center I thro' H ſhall paſs thro' H, E, F, G, and touch the ſides of the ſquare, being the angles H, E, F, G, are right. *Which was to be done.*

PROP.

PROP. IX.



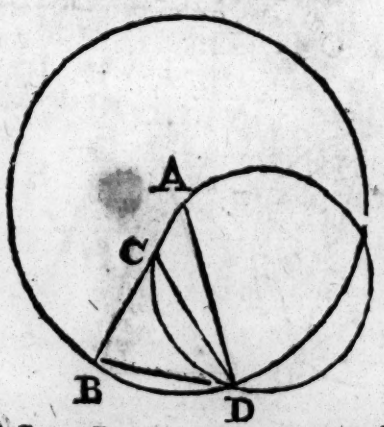
About a ſquare given ABCD to deſcribe a circle EABCD

Draw the diameters AC, BD cutting one the other in E. From the center E thro' A deſcribe a circle, then I ſay that circle is deſcribed about the ſquare.

For the angles ABD and BAC are *a* half of right angles, *b* therefore EA = EB. After the ſame manner is EA = ED = EC. The circle therefore deſcribed from the center E paſſes thro' A, B, C, D the angles of the ſquare given. *Which was to be done.*

a 4. *Cor.* 33. 1.
b 6. 4.

PROP. X.



To make an Iſoſceles triangle A B D, having each angle at the baſe B and A D B double to the remaining angle A.

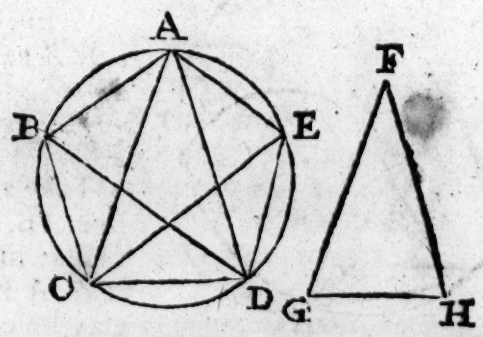
Take any right line AB, and divide it in C, *a* ſo that A B × B C may be equal to ACq. From the center A thro' B, deſcribe the circle ABD; and in this circle *b* apply BD = AC, and join AD; I ſay ABD is the triangle required.

a 11. 2.
b 1. 4.

For, draw DC, and thro' the points C, D, A.
e 5. 4. *e* draw a circle. Now becauſe AB × BC = ACq, *d*
d 37. 3. it is evident that FD touches the circle ACD
e 32. 3. which CD cutteth; *e* therefore is the angle BDC
f 2. *ax.* = A, and therefore the angle BDC + CDA *f* =
g 32. 1. A + CDA *g* = BCD. But BDC + CDA = BDA
h 5. 1. *h* = CBD. *k* therefore the angle BCD = CBD,
k 1. *ax.* and therefore DC *l* = DB = *m* AC, *n* wherefore
l 6. 1. the angle CDA = A = BDC. therefore ADB
m *conſtr.* = 2 A = ABD. *Which was to be done.*
n 5. 1. This conſtruction is Analytically found out thus; take the thing for done, and let the right line DC biſect the angle BDA; *a* therefore DA,
a 3. 6. DB :: CA, CB. alſo becauſe the angle CDA *b* =
b *conſtr.* ½ ADB *c* = A, *c* therefore CA = DC. and be-
c *hyp.* cauſe the angle DCB *b* = A + CDA = 2 A *e* =
d 6. 1. B, *d* thence will be DB = DC. *f* from whence
e 31. 1. alſo DB = CA. and ſo DA (BA.) CA :: CA. CB. *g*
f 2. *ax.* whence BA × CB = CAq.
g 17. 6. *Coroll.*

Whereas all the angles A, B, D *h* make up
h 32. 1. two right angles, it's evident that A is ⅓ of two right angles.

PROP. XI.



In a circle given ABCDE to deſcribe a Pentagone figure ABCDE equilateral and equiangular.

a Deſcribe an Iſoſceles triangle FGH, having
a 10. 4. each angle at the baſe double to the other; *b* in-
b 2. 4. ſcribe

ſcribe a triangle CAD equiangular to the ſaid triangle FGH. *c* Biſect the angles at the baſe ACD and ADC with the right lines DB, CE meeting with the circumference in B and E. join the right line CB, BA, AE, ED. *Then I ſay it is done.* c 9. 1.

For it is evident by conſtruction that the angles CAD, CDB, BDA, DCE, ECA, are equal; wherefore the *d* arches and *e* ſubtended lines DC, CB, BA, AE, DE are equal. Therefore the Pentagone is equilateral, and equiangular *f* becauſe the angles of it BAE, AED, *&c.* ſtand on equal *g* arches BCDE, ABCD, *&c.* d 26. 3. e 29. 3. f 27. 3. g 2. *ax.*

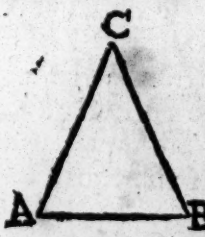
A more eaſy Practice of this Problem ſhall be deliver'd at 10, 13.

Coroll.

Hence, each angle of an equilateral and equiangular Pentagone is equal to $\frac{3}{5}$ of two right angles, or $\frac{6}{5}$ of one right angle.

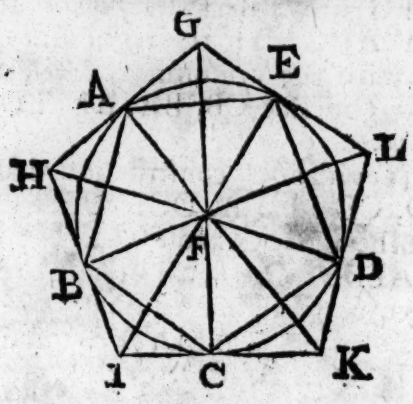
Schol.

Generally all figures of odd number of ſides are inſcribed in circles by the help of Iſoſceles triangles, whoſe angles at the baſe are multiples of thoſe at the top: and figures of even number of ſides are inſcribed in a circle by the help of Iſoſceles triangle, whoſe angles at the baſe are multiples ſeſquialter of thoſe at the top. Pet. Herig.



As in the Iſoſceles triangle CAB, if the angle A = 3 C = B, then will AB be the ſide of a Heptagone. If A = 4 C, then is AB the ſide of Enneagone. But if A = $\frac{1}{2}$ C, then is AB the ſide of a ſquare. And if A = 2 $\frac{1}{2}$ C CAB will ſubtend the ſixth part of a circumference: and likewiſe if A = 3 $\frac{1}{2}$, then will AB be the ſide of Octagone.

PROP. XII.



About a circle given FABCDE, to deſcribe an equilateral and an equiangular pentagone HIKLG.

a 11. 4. *a* Inſcribe a pentagone ABCDE in the circle given; and from the center draw the right lines FA, FB, FC, FD, FE; and to thoſe lines draw ſo many perpendiculars GAH, HBI, ICK, KDL, LEG meeting in the points H, I, K, L, G. then I ſay it is done. For becauſe GA, GE from the ſame point G *b* touch the circle, *c* therefore is GA = GE, and *d* therefore the angle GFA = GFE, therefore the angle AFE = 2 GFA. After the ſame manner is the angle AFH = FHB, and conſequently the angle AFB = 2 AFH. *e* But the angle AFE = AFB, *f* therefore the angle GFA = AFH. But alſo the angle FAH *g* = FAG, and the ſide FA is common, *h* therefore HA = AG GE = EL, *&c.* *k* Therefore HG, GL, LK, KI, IH the ſides of the pentagone are equal, the angles alſo are, becauſe double of the equal angles AGF, AHF. therefore, *&c.*

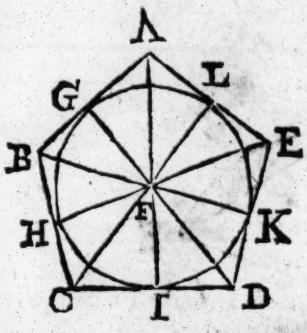
b cor. 16. 3.
c 2. cor. 36. 3.
d 8. 1.
e 27. 3.
f 7. *ax.*
g 12. *ax.*
h 26. 1.
k 3. *ax.*

Coroll.

After the ſame manner, if any equilateral and equiangled figure be deſcribed in a circle, and at the extreme points of the ſemi-diameters drawn from the center to the angles, be drawn perpendicular lines to the ſaid diameters, I ſay that theſe per-

perpendiculars ſhall make another figure of as many equal ſides and equal angles, deſcribed about the circle.

PROP. XIII.



In an equilateral and equiangular pentagone given ABCDE, to inſcribe a circle FGHK.

a Biſect two angles of the pentagone A and B with the right lines AF, BF meeting in the point F. From F draw the perpendiculars FG, FH, FI, FK, FL. Then a circle deſcribed from the center F thro' G will touch all the ſides of the pentagone. a 9. 1.

Draw FC, FD, FE. Becauſe BA *b* = BC, and the ſide BF common, and the angle FBA *c* = FBC, *d* therefore is AF = FC and the angle FAB = FCB, but the angle FAB *e* = ½ BAE = ½ BCD. Therefore the angle FCB = ½ BCD. After the ſame manner are all the whole angles C, D, E biſected. Now whereas the angle FGB *f* = FHB, and the angles FBH = FBG and the ſide FB is common *g* therefore is FG = FH. In like manner are all the right lines FH, FI, FK, FL, FG equal. Therefore a circle deſcribed from the center F thro' G paſſes thro' the points H, I, K, L and *h* touches the ſides of the pentagone, becauſe the angles at thoſe points are right. *Which was to be done.*

b *hyp.* c *conſtr.* d 4. 1. e *hyp.* f 12. *ax.* g 26. 1. h *cor.* 16. 3.

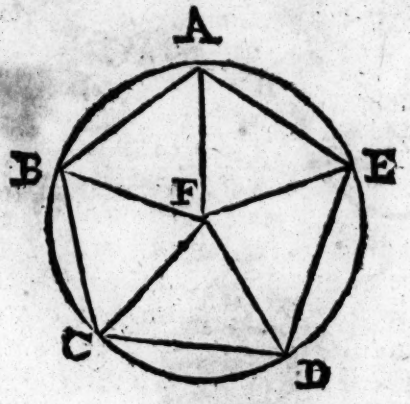
Coroll.

Hence, If any two neareſt angles of an equilateral and equiangular figure be biſected, and from that point in which the lines meet that biſect the angles be drawn right lines to the remaining angles of the figure, all the angles of the figure ſhall be biſected.

Schol.

By the ſame method ſhall a circle be inſcribed in any equilateral or equiangular figure.

PROP. XIV.



About a pentagone given ABCDE equilateral and equiangular to deſcribe a circle FABCDE.

Biſect any two angles of the pentagone with the right lines AF, BF meeting in the point F; the circle deſcribed from the center F thro' A ſhall be deſcribed about the pentagone.

For let FC, FD, FE be drawn. *a* Then the angles C, D, E are biſected; *b* and therefore FA, FB, FC, FD, FE are equal; therefore the circle deſcribed from the center F paſſes thro' A, B, C, D, E all the angles of the pentagone. *Which was to be dem.*

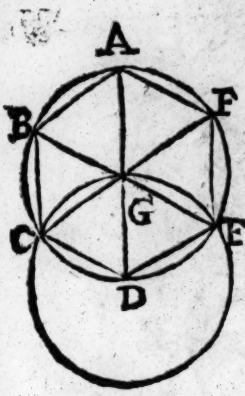
a cor. 13. 4.
b 6. 1.

Schol.

By the ſame art is a circle deſcribed about any figure which is equilateral and equiangular.

PROP.

PROP. XV.



In a circle given GABCDEF to inſcribe an Hexagone (or ſix-ſided figure) equilateral and equiangular ABC DEF.

Draw the diameter AD; from the center D through the center G deſcribe a circle cutting the circle given in the points C and E. Draw the diameters CF, EB; and join AB, BC, CD, DE, EF, FA. *Then I ſay it's done.*

For the angle CGD *a* = $\frac{1}{3}$ of 2 right *a* = DGE *b* = AGF *b* = AGB. Therefore BGC = $\frac{1}{3}$ of 2 right = FGE; therefore the *d* arches and *e* ſubtenſes AB, BC, CD, DE, EF are equal. Therefore the Hexagone is equilateral, but it is equiangled alſo, *f* becauſe all the angles of it ſtand upon equal arches.

a 32. 1.
b 15. 1.
c *cor.* 13. 1.
d 26. 3.
e 29. 3.
f 27. 3.

Coroll.

1. Hence, The ſide of an Hexagone inſcribed in a circle is equal to the ſemidiameter.
2. Hereby an equilateral triangle ACE may very eaſily be deſcribed in a circle given.

Schol.

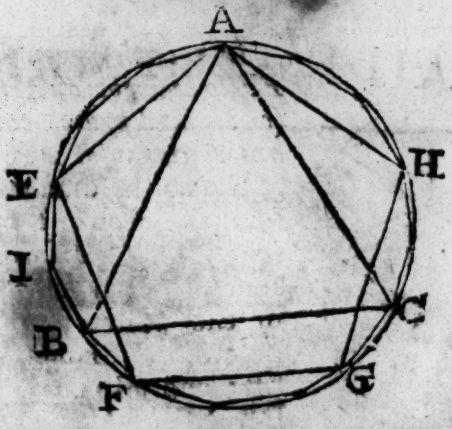
To make a true Hexagone upon a right line given CD.

a Make an equilateral triangle CGD upon the line given CD; from the center G thro' C and D deſcribe a circle. That circle ſhall contain the Hexagone made upon the given line CD.

Andr. *Tacq.* a 1. 1.

PROP.

PROP. XVI.



In a circle given AEBC to inſcribe a quindecagone (or fifteen-ſided figure) equilateral and equiangular.

a 11. 4. b 3. 4. *a* Inſcribe an equilateral pentagone AEFGH in the circle given, *and b* alſo an equilateral triangle ABC. then I ſay BF is the ſide of the quindecagone required.

c *conſtr.* For the arch AB *c* is $\frac{1}{3}$ or $\frac{5}{15}$ of that periphery whereof AF is $\frac{1}{5}$ or $\frac{3}{15}$. therefore the remaining part BF is $\frac{6}{15}$ of the periphery; and therefore the quindecagone, whoſe ſide is BF, is equilateral;

d 27. 3. but it is equiangular alſo, *d* becauſe all the angles inſiſt on equal arches of a circle, where every one $\frac{1}{15}$ of the whole circumference. Therefore, *&c.*

Schol.

A circle is geometrically divided into parts.
- 4, 8, 16, *&c.* by 6, 4, and 9, 1.
- 3, 6, 12, *&c.* by 15, 4, and 9, 1.
- 5, 10, 20, *&c.* by 11, 4, and 9, 1.
- 15, 30, 60, *&c.* by 16, 4, and 9, 1.

Any other way of dividing the circumference into any parts given, is as yet unknown, wherefore in the conſtruction of ordinate figures, we are forced to have recourſe to mechanick artifices, concerning which you may conſult the Writers of practical Geometry.

THE

THE FIFTH BOOK OF EUCLIDE's ELEMENTS.

Definitions.

I. A Part, is a magnitude of a magnitude, a leſs of a greater, when the leſs meaſures the greater.

II. Multiple is a greater magnitude in reſpect of a leſſer, when the leſſer meaſures the greater.

III. Ratio (or rate) is the mutual habitude or reſpect of two magnitudes of the ſame kind each to other, according to quantity.

In every ratio, that quantity which is referred to another quantity is called the antecedent of the ratio, and that to which the other is referred is called the conſequent of the ratio, as in the ratio of 6 to 4, 6 is the antecedent, and 4 the conſequent.

Note, *The quantity of any ratio is known by dividing the antecedent by the conſequent; as the ratio of 12 to 5 is expreſſed by* $\frac{12}{5}$, *or the quantity of the ratio of* A *to* B *is* $\frac{A}{B}$. *Wherefore, often for brevity ſake, we denote the quantities of rations thus;* $\frac{A}{B} \sqsubset$, or $=$, or $\sqsupset \frac{C}{D}$. *that is, the ratio of A to B is greater, equal, or leſs than the ratio of C to D. And this note muſt be diligently obſerved in the underſtanding of the following Book.*

Concerning the divers ſpecies of ratio's, you may pleaſe to conſult Interpreters.

IV. Proportion is a ſimilitude of ratio's.

That which is here termed Proportion, *is more rightly called* Proportionality *or* Analogy; *for pro-*

proportion commonly denotes no more than the ratio betwixt two magnitudes.

V. Thoſe numbers are ſaid to have a ratio betwixt them, which being multiplied may exceed one the other.

E,12 | A, 5. B,6. | G,24.
F,30. | C,10. D,15. | H,60.

VI. Magnitudes are ſaid to be in the ſame ratio, the firſt A to the ſecond C, and the third C to the fourth D, when the equimultiples E and F of the firſt A, and the third C compared with the equimultiples G, H of the ſecond B and the fourth D, according to any multiplication whatſoever, either both together E, F are leſs than G, H both together, or equal taken together, or exceed one the other together, if thoſe be taken E, G and F, H, which anſwer one to the other.

The note hereof is : : ; *as A. B* : : *C. D. That is, as A is to B, ſo is C to D. which ſignifies that A to B, and C to D are in the ſame ratio. We ſometimes thus expreſs it;* $\frac{A}{C} = \frac{C}{D}$ *that is A. B* : : *C. D.*

VII. Magnitudes that have the ſame ratio (A. B : : C. D. are called proportional.

E,30. | A,6. B,4. | G,28.
F,60. | C,12. D,9. | H,63.

VIII. When of equimultiples, E the multiple of the firſt magnitude A exceeds G the multiple of the ſecond B, but F the multiple of the third C exceeds not H the multiple of the fourth D, then the firſt A to the ſecond B has a greater ratio than the third C to the fourth D.

If $\frac{A}{B} \sqsubset \frac{C}{D}$, *it is not neceſſary from this definition that E ſhould always exceed G, when F is leſs than H; but it is granted that this may be.*

IX. Proportionality conſiſts in three terms at leaſt. *Whereof the ſecond ſupplies the place of two.*

X. When 3 magnitudes A, B, C are proportional,

nal, the first A shall have a duplicate ratio to the third C of that it has to the second B: But when 4 magnitudes A,B,C,D are proportional, the first A shall have a triplicate ratio to the fourth D of what it had to the second B; and so always in order one more, as the proportion shall be extended.

Duplicate ratio is thus expressed $\frac{A}{C} = \frac{A}{B}$ *twice, that is, the ratio of A to C is double of the ratio of A to B.*

Triple ratio is thus expressed; $\frac{A}{D} = \frac{A}{B}$ *thrice. That is the ratio of A to D is triple of the ratio of A to B.*

$\div\div$ *denotes continual proportionals*; as A,B,C,D; or 2, 6, 18, 64, *are* $\div\div$.

XI. Magnitudes of a like ratio, are antecedents to antecedents, and consequents to consequents; *As if A.B :: C.D. A and C; and B and D are homologous or magnitudes of a like ratio.*

XII. Alternate proportion is the comparing of antecedent to antecedent, and consequent to consequent. *As if A.B :: C.D. therefore alternately, or by permutation, A.C :: B.D. by the* 16. *of* 5.

In this definition, and the 5 *following, names are given to the six ways of arguing which are often used by Mathematicians: the force of which inferences depends on the Propositions of this Book, which are named in their Explications.*

XIII. Inverse ratio is when the consequent is taken as the antecedent, and so compared to the antecedent as the consequent; *As A. B :: C. D. therefore generally B. A :: D. C. by cor.* 4. 5.

XIV. Compounded ratio is when the antecedent and consequent taken both as one are compared to the consequent it self. *As A. B :: C. D. therefore by composition A + B. B :: C + D. D by* 18. 5.

XV. Divided ratio is when the excess wherein the antecedent exceeds the consequent, is compared to the consequent. *As A.B :: C.D. therefore by division A − B. B :: C − D. D. by* 17. 5.

XVI.

XVI. Converſe ratio is when the antecedent is compared to the exceſs wherein the antecedent exceeds the conſequent. *As A. B :: C. D. therefore by converſe ratio A. A−B :: C. C−D. by the coroll. of the* 19. *of the* 5.

XVII. Proportion of equality is where there are taken more magnitudes than two in one order, and alſo as many magnitudes in another order, comparing two to two being in the ſame ratio; it comes to paſs that as in the firſt order of magnitudes, the firſt is to the laſt, ſo in the ſecond order of magnitudes is the firſt to the laſt. Or otherwiſe: it is a compariſon of the extremes together, the mean magnitudes being taken away.

XVIII. Ordinate proportionality is, when as the antecedent is to the conſequent, ſo is the antecedent to the conſequent, and as the conſequent is to any other, ſo is the conſequent to any other. *As A.B :: D. E. alſo B.C :: E.F. it ſhall be true alſo A. C :: D. F. by the* 22. *of the* 5.

XIX. Inordinate proportion is, when three magnitudes being put, and others alſo, which are equal to theſe in multitude, as in the firſt magnitudes the antecedent is to the conſequent, ſo in the ſecond magnitude is the antecedent to the conſequent; and as in the firſt magnitudes the conſequent is to any other, ſo in the ſecond magnitudes any other thing to the antecedent. *As A. B :: F. G. alſo B.C :: E.F. it ſhall be true in inordinate proportion. A.C :: E.G. by the* 23. *of the* 5.

XX. Any number of magnitudes being put; the proportion of the firſt to the laſt is compounded out of the proportions of the firſt to the ſecond, the ſecond to the third, and the third to the fourth, and ſo forwards till the proportion ariſe.

Let there be any number of magnitudes A,B,C, D. by this definition $\frac{A}{D} = \frac{A}{B} + \frac{B}{C} + \frac{C}{D}$.

Axioms.

Axiome.

Magnitudes equimultiples to the ſame multiple, are alſo equimultiples betwixt themſelves.

PROP. I.

If there be a number of magnitudes how many ſoever, AB, CD equimultiples to a like number of magnitudes E, F each to other; how multiple one magnitude AB is one E, ſo multiples are all the magnitudes AB + CD to all the other magnitudes E + F.

Let AG, GH, HB the parts of the quantity AB, be equal to E, and alſo let CI, IK, KD the parts of the quantity CD be equal to F. The number of theſe are put equal to thoſe. Now whereas AG + CI *a* = E + F; *a* and GH + IK = E + F; *a* and HB + KD = E + F; it is evident that AB + CD does ſo often contain E + F as one AB contains E. *Which was to be done.* *a* 2. *ax.*

PROP. II.

If the firſt AB be equimultiple to the ſecond C, as the third DE is to the fourth F, and if the fifth BG be equimultiple to the ſecond C as the ſixth EH is to the fourth F; then ſhall the firſt compounded with the fifth (AG) be equimultiple to the ſecond C, as the third compounded with the ſixth (DH) is to the fourth F.

The number of parts in AB equal each to C is put equal to the number of parts in DE. whereof each part is equal to F. Likewiſe the number of parts in BG is equal to the number of parts in EH. Therefore the number of parts in AB + BG is equal to the number of parts in DE + EH.

a 2. *ax.* FH. *a* That is, the whole line AG is as equimultiple of C, as the whole line DH is of F. *Which was to be dem.*

PROP. III.

If the first A be equimultiple of the second B, and the third C of the fourth D, and there be taken EI, FM equimultiples of the first and third, then will each of the magnitudes taken be alike equimultiple of both, the one EI to the second B, the other FM to the fourth D.

Let EG, GH, HI the parts of the multiple EI be equal to A, also let FK, KL, LM the parts of the multiple FM be equal to F, *a* the number of these is equal to the number of those. Moreover A (that is) EG or GH, or HI is put as equimultiple of B, as C, or FK, *&c.* of D. *b* Therefore EG + GH is equimultiple of the second B, as FK + KL is of the fourth D. *c* By the same way of argument is EI (EH + HI) as multiple of B, as FM (FL + LM) is of D. *Which was to be done.*

a *hyp.*

b 205.

c 2. 5,

PROP.

PROP. IV.

If the first A have the same ratio to the second B, as the third C to the fourth D; then also E and F the equimultiples of the first A and the third C, shall have the same ratio to G and H the equimultiples of the second B and the fourth D, according to any multiplication, if so taken as they answer each to other (E. G :: F. H.)

Take I and K the equimultiples of E and F; and also L and M the equimultiples of G and H. *a* Then is I as multiple of A, as K of C; *a* and also L is as multiple of B, as M of D. Therefore whereas it is A. B :: C. D; according to the sixth definition, if I be ⊏, =, ⊐ L, then consequently after the same manner is K ⊏, =, ⊐, M. Therefore when I and K are taken as multiples of E and F, as L and M of G, and , then will it be by the seventh definition E. G :: F. H. *Which was to be dem.*

a 3. 5.

b *hyp.*

Coroll.

From hence is wont to be demonstrated the proof inverse ratio.

For because A. B :: C. D, therefore if E ⊏, =, ⊐ G, then is *c* likewise F ⊏, =, ⊐ H. *erefore* it is evident that if G ⊏, =, ⊐ E, *en* is H ⊏, =, ⊐ F; *d* therefore B. A :: D. C. *hich was to be dem.*

c 2. *def.* 5.

d 6. *def.* 5.

PROP. V.

If a magnitude A be as multiple of magnitude CD, as part taken from the o AE of a part taken from the other CF; the resid of the one shall be as multiple of the residue of t other FD as the whole AB is of the whole CD.

Take any other GA, which shall be as mu tiple to FD the residue, as AB is of the who CD, or as the part taken away AE is of the pa
a 1. 5. taken away CF. *a* Therefore the whole GA + AE is as multiple of the whole CF + FD, the one AE is of the one CF, that is as A
b 6. *ax.* is of CD. therefore GE *b* = AB; and *c* so A
c 3. *ax.* that was common being taken away, there re mains GA = EB.

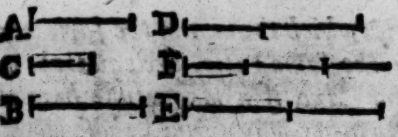
PROP. VI.

If two magnitudes AB, CD be equ multiples of two magnitudes E, F and some magnitudes AG and CH equimultiples of the same E, F, be ta ken away; then the residues GB, HD are either equal to those magnitud E, F, or else equimultiples of them.

For because the number of par in AB, whereof each is equal to E is put equal to the number of par in CD, whereof each is equal t F, and also the number of parts AG equal to the number of par in CH; If from one you take AG, and from th
a 3. *ax.* 3. other CH, *a* then remains the number of par in the remainder GB equal to the number parts in HD. therefore if GB be once E, then HD once F. if GB be many times F, then is HD so of F. *Which was to be dem.*

PRO

PROP. VII.



Equal magnitudes A and B have to the ſame magnitude C the ſame proportion or ratio. And one and the ſame magnitude C has the ſame ratio to equal magnitudes A and B.

Take D and E equimultiples of the equal magnitudes A and B, and F any-wiſe multiple of C; then is D *a* = E. Wherefore if D ⌐, =, ¬ F, then alſo E will be ⌐, =, ¬ F. *b* therefore A. C :: B. C. and *c* by inverſion C. A :: *c* C. B. *Which was to be dem.*

a 6. *ax.*
b 6. *def.* 5.
c *cor.* 4. 5

Schol.

If inſtead of the multiple F, two equimultiples be taken, it ſhall be the ſame way prov'd that equal magnitudes have the ſame ratio to other magnitudes that are equal between themſelves.

PROP. VIII.

Of unequal magnitudes AB, AC, the greater AB has a greater ratio to the ſame third line D, than the leſſer AC; and the ſame third line D hath a greater ratio to the leſſer AC, than to the greater AB.

Take EF, EG equimultiples of the ſaid AB, AC, ſo that EH being multiple of D be greater than EG, but leſſer than EF. (which will eaſily happen, if both EG and GF be taken greater than D.) It is manifeſt from 8. *def.* 5. that $\frac{AB}{D} \sqsubset \frac{AC}{D}$ and $\frac{D}{AB} \sqsupset \frac{D}{AC}$. *Which was to be dem.*

PROP. IX.

Magnitudes which to one and the same magnitude have the same ratio, are equal the one to the other. And if a magnitude have the same ratio to other magnitudes, those magnitudes are equal one to the other.

1. *Hyp.* If A.C :: B.C; I say that A=B,
a 8. 5. For let A be greater or less than C. *a* then is $\frac{A}{C} \sqsubset$ or $\sqsupset \frac{B}{C}$. *Which is contrary to the Hypothesis.*

2. *Hyp.* If C. B :: C. A. I say that A=B. For
b 8. 5. let A be $\sqsubset$ B, *b* then $\frac{C}{B} \sqsubset \frac{C}{A}$. *Which is against the Hypothesis.*

PROP. X.

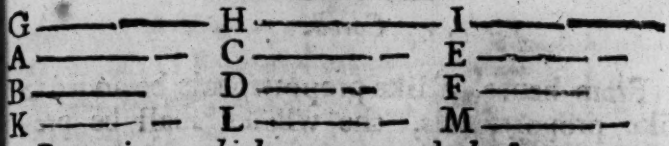
Of magnitudes having ratio to the same magnitude, that which has the greater ratio, is the greater magnitude; and that magnitude to which the same carries a greater ratio is the lesser magnitude.

1. *Hyp.* If $\frac{A}{C} \sqsubset \frac{B}{C}$. I say that A $\sqsubset$ B
a 7. 5. For if it be said that A=B, *a* then A. C :: B.C.
b 8. 5. *Which is contrary to the Hyp.* If A$\sqsupset$B, *b* then is $\frac{A}{C} \sqsupset \frac{B}{C}$. *Which is also against the Hyp.*

2. *Hyp.* If $\frac{C}{B} \sqsubset \frac{C}{A}$ I say that B$\sqsupset$A. for if you say B=A, it's against the Hypothesis, for it will
c 7. 5. *c* follow that C. B :: C. A. If you say B $\sqsubset$ A, *d*
d 8. 1. then is $\frac{C}{A} \sqsubset \frac{C}{B}$. *Which is also against the Hyp.*

PROP.

PROP. XI.



Proportions which are one and the ſame to any third, are alſo the ſame one to another.

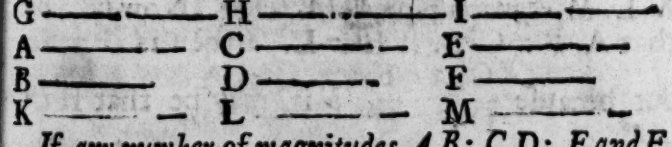
Let A. B :: E. F, and C. D :: E. F. I ſay that A. B :: C. D. Take G, H, I the equimultiples of A, C, E; and K, L, M the equimultiples of B, D, F. Now *a* becauſe A. B :: E. F, if G ⊏, =, ⊐ K, *b* then after the ſame manner I ⊏, =, ⊐ M. And likewiſe *a* becauſe E. F :: C. D. if I ⊏, =, ⊐ M, *b* then is H likewiſe ⊏, =, ⊐ L. *c* wherefore A. B :: C. D. *Which was to be dem.*

a *hyp.*
b 6. *def.* 3.
c 6. *def.* 5.

Schol.

Proportions that are one and the ſame to the ſame proportions, are the ſame betwixt themſelves.

PROP. XII.



If any number of magnitudes A, B; C, D; E and F be proportionals; as one of the Antecedents A is to one of the Conſequents B; ſo are all the Antecedents A, C, E to all the Conſequents B, D, F.

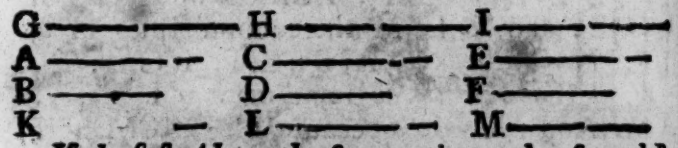
Take the equimultiples of the antecedents G, H, I, and of the conſequents K, L, M. Becauſe that as multiple as one G is of one A, *a* ſo multiples are all G, H, I, of all A, C, E; and likewiſe as multiple as one K is of one B, ſo multiples are all K, L, M, of all B, D, F. moreover becauſe A. B *b* :: C. D *b* :: E. F. if G be ⊏, =, or ⊐ K. then will H likewiſe be ⊏, =, ⊐ L, and I ⊏, =, ⊐ M. and ſo if G ⊏, =, ⊐ K. in like manner will G + H + I be ⊏, =, ⊐ K + L + M. *c* wherefore A. B :: A + C + E. B + D + F. *Which was to be dem.*

a 1. 5.
b *hyp.*
c 6. *def.* 5.

Coroll.

From hence, if like proportionals be added to like proportionals, the wholes ſhall be proportional.

PROP. XIII.



If the firſt A have the ſame ratio to the ſecond B, that the third C has to the fourth D; and if the third C have a greater proportion to the fourth D, than the fifth E to the ſixth F; then alſo ſhall the firſt A have a greater proportion to the ſecond B, than the fifth E to the ſixth F.

Take G, H, I equimultiples of A, C, E, and K, L, M, equimultiples of B, D, F. Now becauſe
a 6. *def.* 1. that A. B :: C. D. if H ⊏ L. *a* then is G ⊏ K.
b 8. *def.* 5. but becauſe $\frac{C}{D} \sqsubset \frac{E}{F}$, *b* it may be that H ⊏
c 8. *def.* 5. L, and yet I not ⊏ M. *c* Therefore $\frac{A}{B} \sqsubset \frac{E}{F}$.

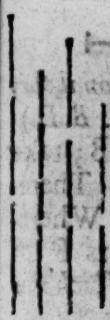
Which was to be dem.

Schol.

But if $\frac{C}{D} \sqsupset \frac{E}{F}$, then alſo is $\frac{A}{B} \sqsupset \frac{E}{F}$. Alſo, if $\frac{A}{B} \sqsubset \frac{C}{D} \sqsubset \frac{E}{F}$, then is $\frac{A}{B} \sqsubset \frac{E}{F}$. And if $\frac{A}{B} = \frac{C}{D} \sqsupset \frac{E}{F}$, then is $\frac{A}{B} \sqsupset \frac{E}{F}$.

PROP.

PROP. XIV.



If the first A have the same ratio to the second B, that the third C has to the fourth D; and if the first A be greater than the third C; then shall the second B be greater than the fourth D. But if the first A be equal to the third C, then the second B shall be equal to the fourth D. but if A be lesser, the is B also lesser.

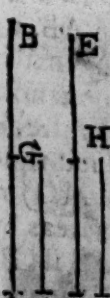
Let A ⊏ C *a* then $\frac{A}{B} \sqsubset \frac{C}{B}$. *b* but $\frac{A}{B} = \frac{C}{D}$. *c* therefore $\frac{C}{D} \sqsubset \frac{C}{B}$. *c* therefore B ⊏ D. By the like way of argument, if A ⊐ C, *d* then is B ⊐ D. But if A be put equal to C, then C. B :: *e* A. B *f* :: C. D. *g* therefore B = D. *Which was to be dem.*

a 8. 5.
b *hyp.*
c 13. 5.
d 10. 5.
e 7. 5.
f *hyp.*
g 11. 5. & 9. 5.

Schol.

By an argument *à fortiori*, if $\frac{A}{B} \sqsupset \frac{C}{D}$, and A ⊏ C. then is B ⊏ D. Likewise if A = B, then is B = D. and if A ⊏, or ⊐ B, then also is C ⊏ or ⊐ D.

PROP. XV.



Parts C and F are in the same ratio, with their like multiples AB and DE, if taken correspondently. (*AB. DE :: C. F.*)

Let AG, GB parts of the multiple AB be equal to C; and let DH, HE parts of the multiple DE be equal to F. *a* The number of these parts is equal to the number of those. Therefore whereas *b* AG, C :: DH. F, and GB. C :: HE. F, therefore is *c* AG + GB (AB.) DH + HE (DE) :. C. F. *Which was to be dem.*

a *hyp.*
b 7. 5.
c 12. 5.

PROP. XVI.

If four magnitudes A, B, C, D be proportional, they also shall be alternately proportional (A. C :: B. D.)

Take E and F equimultiples of A and B; take also G and H equimultiples of C and D. Therefore E. F *a* :: A. B *b* :: C. D *a* :: G. H. Wherefore if E ⊏, —, ⊐ G, *c* then likewise is F ⊏, =, ⊐ H. *d* Therefore A. C :: B. D. *Which was to be dem.*

a 15. 5.
b *hyp.*
c 11. 5. & 14. 5.
d 6. *def.* 5.

Schol.

Alternate ratio has place only then when the quantities are of the same kind. For heterogeneous quantities are not compared together.

PROP. XVII.

If magnitudes compounded be proportional AB. CB :: DE. FE.) they shall be proportional also when divided (AC. CB :: DF. FE.

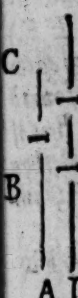
Take GH, HL, IK, KM, in order the equimultiples of AC, CB, DF, FE; and also LN, MO, the equimultiples of CB, FE. The whole GL is *a* as multiple of the whole AB, as one GH of one AC, *b* that is as IK of DF, *c* or as the whole IM of the whole DE. Also HN (HL + LN) is as *d* multiple of CB, as KO (KM + MO) is of FE. Therefore, whereas by Hyp. AB. BC :: DE. EF. if GL be ⊏, =, ⊐ HN, then likewise *e* will IM ⊏, =, ⊐ KO. Take from these HL, KM that are equal;

a 1. 5.
b *constr.*
c 1. 5.
d 2. 5.
e 6. *def.*

equal; and if the remainder GH be ⊏, =, ⊐ LN, *f* then will IK ⊏, —, ⊐ MO, *g* whence AC. CB :: DF. EF. *Which was to be dem.*

f 5. *ax.*
g 6. *def.* 5.

PROP. XVIII.

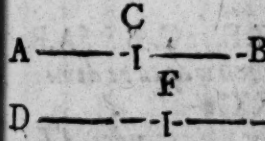


If magnitudes divided be proportional (AB. BC :: DE. EF) the same also being compounded shall be proportional (AC. CB :: DF. FE.

For if it can be, let AB. CB :: DF. FG ⊐ FE. *a* Then by division will AB. BC :: DG. GF. *b* that is, DG, GF :: DE. EF. and being DG = DE, *c* therefore is GF ⊏ EF. *d Which is absurd.* The like absurdity will follow if it be said AB. CB :: DE. GF ⊏ FE.

a 17. 5.
b *hyp.* & 11. 5.
c 14. 5.
d 9. *ax.*

PROP. XIX.



If the whole AB be to the whole DE as the part taken away AC is to the part taken away DF, then shall the residue CB be to the residue FE as the whole AB is to the whole DE.

Because *a* AB. DE :: AC. DF, *b* therefore by permutation AB. AC :: DE. DF. *c* and thence by division AC. CB :: DF. FE. *b* wherefore again by permutation AC. DF :: CB. FE. *d* that is, AB. DE :: CB. FE. *Which was to be dem.*

a *hyp.*
b 10. 5.
c 17. 5.
d *hyp.* & 11. 5.

Coroll.

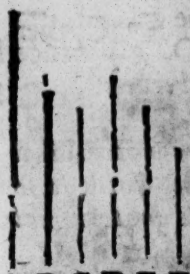
Hence, If like proportionals be subtracted from like proportionals, the residues shall be proportional.

2. *Hence is converse ratio demonstrated.*

Let AB. CB :: DE. FE. I say that AB. AC :: DE. DF. For by *a* permutation AB. DE :: CB. FE. *b* therefore AB. DE :: AC. DF. whence again by permutation AB. AC :: DE. DF. *Which was to be dem.*

a 16. 5.
b 17. 5.

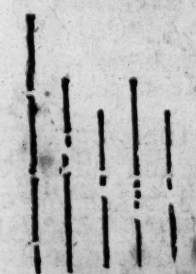
PROP.



If there be three magnitudes A,B,C, and others D, E, F equal to those in number, which being taken two and two in each order are in the same ratio, (A. B :: D. E ; and B.C :: E.F,) and if of equality the first A be greater than the third C ; then shall the fourth D be greater than the sixth F. But if the first A be equal to the third C, then the fourth D is so to the sixth F ; and if A be less than C, so D is less than F.

a *hyp.* 1. *Hyp.* Let A ⊐ C. Because *a* E.F :: B. C. by *b* in-
b *cor.* 4. 5. version shall be F.E :: C.B. *c* But $\frac{C}{B} \sqsupset \frac{A}{B}$, therefore
c *hyp. and* 8. 5.
d *schol.* 13. 5. $\frac{F}{E} \sqsupset \frac{A}{B}$ or $\frac{D}{E}$, *e* therefore D ⊐ F. *W. W. to be dem.*
e 10. 5. 2. *Hyp.* By the same way of argument, if A ⊐ C,
f 7. 5. it will appear that D ⊐ F.
g 11. 5. *and* 9. 5. 3. *Hyp.* If A = C. Because F.E, :: C. B :: *f* A.B :: D.E. *g* therefore is D = F. *Which was to be dem.*

PROP. XXI.



If there be three magnitudes A,B,C, and others also D, E, F equal to them in number, which taken two and two are in the same ratio ; and their proportion inordinate (A. B :: E. F. and B. C :: D.E.) and if of equality, the first A be greater than the third C ; then is the fourth D greater than the sixth F : but if the first be equal to the third, then is the fourth equal to the sixth ; if less, so is the other likewise.

a *hyp.* 1. *Hyp.* If A ⊐ C ; then because *a* D. E :: B. C.
b 8. 5. therefore inversely E.D :: C.B. but $\frac{C}{B}$ *b* $\sqsupset \frac{A}{B}$ *c* there-
c *schol.* 13. 5. fore $\frac{E}{D} \sqsupset \frac{A}{B}$, that is $\frac{E}{F}$, *d* therefore D ⊐ F.
d 10. 5.

2. *Hyp.*

2. *Hyp.* By the like arguments, if A ⊐ C, then is D ⊐ F.

3. *Hyp.* If A = C; then becauſe E. D :: *e* C. B :: *e* A. B :: *f* E. F, *g* therefore is D = F. *Which was to be dem.*

e 7. 5.
f *hyp.*
g 9. 5.

PROP. XXII.

If there be any number of magnitudes A, B, C, and others equal to them in number D, E, F, which taken two and two are in the ſame ratio (A. B :: D. E. and B. C :: E. F) they ſhall be in the ſame ratio alſo by equality, (A. C :: D. F.

Take G. H equimultiples of A, D; and I, K of B, E; and alſo L, M of C, F.

Becauſe *a* A. B :: D. E, *b* therefore G. I :: H. K. and in like manner I. L :: K. M. therefore, if G ⊏, =, ⊐, L, *c* then is H ⊏, =, ⊐ M, *d* therefore A. C :: D. F. By the ſame way of demonſtration if further C. N :: F. O, then by equality A. N :: D. O. *Which was to be dem.*

a *hyp.*
b 4. 5.
c 20. 5.
d 6. *def.* 5.

PROP.

PROP. XXIII.

If there be three magnitudes A,B,C, and others D, E, F, equal to them in number, which taken two and two are in the same ratio, and their proportionality inordinate (A.B :: E.F. and B.C :: D.E.) they shall be in the same ratio also by equality.

Take G, H, I equimultiples of A,B,D; and also K,L, M equimultiples of C,E,F. Then G.H :: *a* A.B :: *b* E.F *a* :: L.M. Moreover because *b* B. C :: D.E, thence is *c* H. K :: I.L; therefore G,H,K, and I,L,M are according to 21.5. Wherefore if G be ⊏, =, ⊐ K, then is likewise I ⊏, =, ⊐ M. and so *d* consequently A. C :: D. F. *Which was to be demonstrated.*

a 15. 5.
b hyp.
c 4. 5.
d 6. *def.* 5.

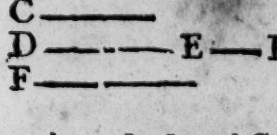
If there be more magnitudes than three, this way of demonstration holds good in them also.

Coroll.

From hence * it follows that ratio's compounded of the same ratio's are among themselves the same; as also the same parts of the same ratio's, are among themselves the same.

*22. & 23. 5. and 20. def. 1.

PROP. XXIV.



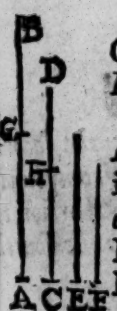
If the first magnitude AB have the same ratio to the second C, which the third DE has to the fourth F; and if the fifth BG have the same ratio to the second C, which the sixth EH has to the fourth F, then shall the first compounded with the fifth (AG) have the same ratio to the second C, which the third compounded with the sixth (DH) has to the fourth F.

For

For because *a* AB.C :: DE. F, and by the Hyp. and inversion C. BG :: F. EH; therefore by *b* equality AB. BG :: DE. EH. whence by compounding AG. BG :: DH. EH. Also *c* BG. C :: EH. F. Therefore again by *b* equality AG. C :: DH. F. *Which was to be dem.*

a *hyp.* b 22. 5. c *hyp.*

PROP. XXV.



If four magnitudes be proportional (AB. CD :: E. F) the greatest AB and the least F shall be greater than the rest CD, and E.

Make AG = E, and CH = F. Because AB. CD :: *a* E. F :: *b* AG. CH. *c* thence is AB. CD :: GB. HD. *d* but AB ⊏ CD. *e* therefore GB ⊏ HD. But AG + F = E + CH, therefore AG + F + GB ⊏ E + CH + HD, that is, AB + F ⊏ E + CD. *Which was to be dem.*

a *hyp.* b 7. 5. c 19. 5. d *hyp.* e *schol.* 14. 5.

These Propositions which follow are not Euclide's, *but taken out of other Authors, and here subjoined because of their frequent use.*

PROP. XXVI.

A——C——
B——D——
E——

If the first have a greater proportion to the second, than the third to the fourth, then contrarywise, by conversion, the second shall have a less proportion to the first, than the fourth to the third.

Let $\frac{A}{B} \sqsubset \frac{C}{D}$. I say that $\frac{B}{A} \sqsupset \frac{D}{C}$. For conceive $\frac{C}{D} = \frac{E}{B}$; *a* therefore $\frac{A}{B} \sqsubset \frac{E}{B}$ *b* whence A ⊏ E. *c* therefore $\frac{B}{A} \sqsupset \frac{B}{E}$, *d* or $\frac{D}{C}$ *Which was to be dem.*

a 13. 5. b 10. 5. c 8. 5. d *cor.* 4. 5.

PROP. XXVII.

A——C——
B——D——
E——

If the first have a greater proportion to the second, than the third to the fourth; then alternately the first shall have a greater proportion to the third, than the second to the fourth.

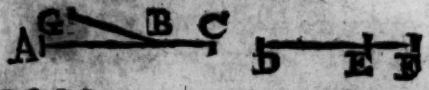
Let

Let $\frac{A}{B} \sqsubset \frac{C}{D}$. then I say $\frac{A}{C} \sqsubset \frac{B}{D}$. For conceiv

$\frac{E}{B} = \frac{C}{D}$. *a* therefore A⊏E, *b* and therefore $\frac{A}{C} \sqsubset \frac{E}{C}$

a 10. 5. *b* 1. 5. *c* 16. 5. *c* or $\frac{B}{D}$. *Which was to be dem.*

PROP. XXVIII.

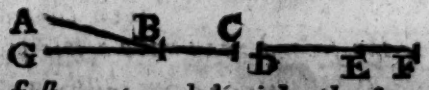


If the first have a greater proportion to the second than the third to the fourth, then the first compounded with the second shall have a greater proportion to the second, than the third compounded with the fourth to the fourth.

Let $\frac{AB}{BC} \sqsubset \frac{DE}{EF}$. I say that $\frac{AC}{BC} \sqsubset \frac{DE}{EF}$. For con-

a 10. 5. *b* 4. *ax.* *c* 8. 5. *d* 18. 5. ceive $\frac{GB}{BC} = \frac{DE}{EF}$. *a* therefore is AB⊏GB. add BC to each part. then *b* will AC ⊏ GC. *c* therefore $\frac{AC}{BC} \sqsubset \frac{GC}{BC}$. *d* that is $\frac{DF}{EF}$. *Which was to be dem.*

PROP. XXIX.



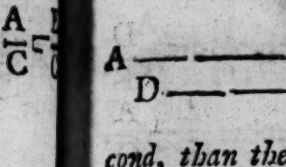
If the first compounded with the second have a greater proportion to the second, than the third compounded with the fourth has to the fourth; then by division the first shall have a greater proportion to the second, than the third to the fourth.

Let $\frac{AC}{BC} \sqsubset \frac{DF}{EF}$. then I say $\frac{AB}{BC} \sqsubset \frac{DE}{EF}$. For con-

a 10. 5. *b* 5. *ax.* *c* 8. 4. *d* 17. 5. ceive $\frac{GC}{BC} = \frac{DE}{EF}$. *a* theref. AC⊏GC. Take away BC, that is common; then *b* remains AB⊏GB. *c* Therefore $\frac{AB}{BC} \sqsubset \frac{GB}{BC}$ *d* or $\frac{DE}{EF}$. *Which was to be dem.*

PROP.

PROP. XXX.



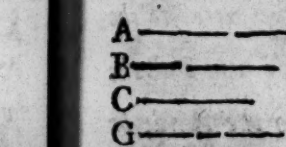
If the first compounded with the second have a greater proportion to the second, than the third compounded with the fourth has to the fourth, then by converse ratio shall the first compounded with the second have a lesser ratio to the first, than the third compounded with the fourth shall have to the third.

Let $\frac{AC}{BC} \sqsubset \frac{DF}{EF}$. Then I say that $\frac{AC}{AB} \sqsupset \frac{DF}{DE}$.

For because that $\frac{AC}{BC}$ *a* $\sqsubset \frac{DF}{EF}$, *b* therefore by division $\frac{AB}{BC} \sqsubset \frac{DE}{EF}$ by conversion *c* therefore $\frac{BC}{AB} \sqsupset \frac{EF}{DE}$ and *d* therefore by composition $\frac{AC}{AB} \sqsupset \frac{DF}{DE}$. *Which was to be dem.*

a *hyp.*
b 29. 5.
c 26. 5.
d 28. 5.

PROP. XXXI.



If there be three magnitudes A, B, C, and others also D, E, F equal to them in number; and if there be a greater proportion of the first of the former to the second, than there is of the first of the last to their second $\left(\frac{A}{B} \sqsubset \frac{D}{F};\right)$ *and there be also a greater proportion of the second of the first magnitudes to the third, than there is of the second of the last magnitude to their third* $\left(\frac{B}{C} \sqsubset \frac{E}{F}.\right)$ *Then by equality also shall the ratio of the first of the former magnitudes to the third be greater than the ratio of the first of the latter magnitudes to the third* $\left(\frac{A}{C} \sqsubset \frac{D}{F}.\right)$

Con-

Conceive $\frac{G}{C} = \frac{E}{F}$. *a* therefore is E ⊏ G, and *b* therefore $\frac{A}{G} \sqsubset \frac{A}{B}$. Again conceive $\frac{H}{G} = \frac{D}{F}$. *c* therefore $\frac{H}{G} \sqsupset \frac{A}{B}$, therefore much more $\frac{H}{G} \sqsupset \frac{A}{G}$. *d* wherefore A ⊏ H, *e* and consequently $\frac{A}{C} \sqsubset \frac{H}{C}$ *f* or $\frac{D}{F}$.

a 10. 5.
b 8. 5.
c 13. 5.
d 10. 5.
e 8. 5.
f 12. 5.

PROP. XXXII.

If there be three magnitudes A, B, C, and others D, E, F, equal to them in number; and there be a greater proportion of the first of the former magnitudes to the second, than there is of the second of the latter to the third $\left(\frac{A}{B} \sqsubset \frac{E}{F}\right)$ *and also the ratio of the second of the former to the third be greater than the ratio of the first of the latter to the second* $\left(\frac{B}{C} \sqsubset \frac{D}{E}.\right)$ *then by equality also shall the proportion of the first of the former to the third, be greater than that of the first of the latter to the third* $\left(\frac{A}{C} \sqsubset \frac{D}{E}.\right)$

The demonstration of this Proposition is altogether like that of the precedent.

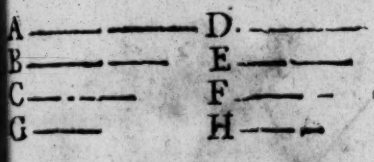
PROP. XXXIII.

If the proportion of the whole AB to the whole CD be greater than the proportion of the part taken away AE to the part taken away CF; then shall also the ratio of the remaindes EB to the remainder FD be greater than that of the whole AB to the whole CD.

Be-

Because that $\frac{AB}{CD}$ *a* $\sqsubset \frac{AE}{EF}$. *b* therefore by permutation $\frac{AB}{AE} \sqsubset \frac{CD}{CF}$, *c* therefore by converse ratio $\frac{AB}{EB} \sqsupset \frac{CD}{FD}$, and by permutation again $\frac{AB}{CD} \sqsupset \frac{EB}{FD}$.

PROP. XXXIV.



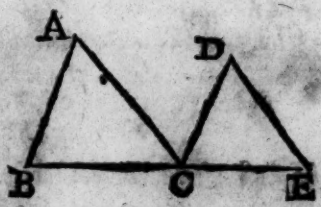
If there be any number of magnitudes, and others also equal to them in number; and the proportion of the first of the former to the first of the latter, be greater than that of the second to the second, and that greater than the proportion of the third to the third, and so forward: all the former magnitudes together shall have a greater ratio to all the latter together, than all the former, leaving out the first, shall have to all the latter, leaving out the first; but less than that of the first of the former to the first of the latter; and lastly greater than that of the last of the former to the last of the latter.

You may please to consult Interpreters for the demonstration hereof, we having for brevity sake omitted it, and because 'tis of no use in these Elements.

The End of the fifth Book.

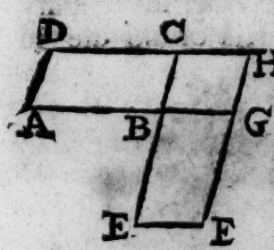
THE SIXTH BOOK OF EUCLIDE's ELEMENTS.

Definitions.

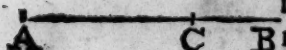


I. LIke right-lined figures (ABC, DCE) are ſuch whoſe ſeveral angles are equal one to the other, and alſo their ſides about the equal angles, proportional.

The angle B = DCE, and AB. BC :: DC. CE. Alſo the angle A = D, and BA. AC :: CD. DE. Laſtly the angle ACB = E, and BC. CA :: CE. ED.

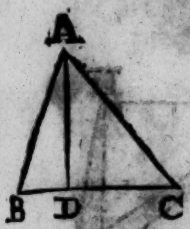


II. Reciprocal figures are (BD, BF) when in either figure are the terms antecedents and conſequents of ratio's, that is, AB. BG :: EB. BC.)



III. A right line AB is ſaid to be cut according to mean and extreme proportion, when as the whole AB is to the greater ſegment AC, ſo is the greater ſegment AC to the leſs CB (AB. AC :: AC. CB.)

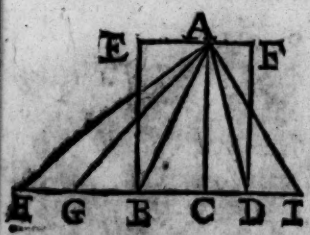
IV. The



IV. The altitude of any figure ABC, is a perpendicular line AD drawn from the top A to the baſe BC.

V. A ratio is ſaid to be compounded of two ratio's, when the quantities of the ratio's being multiplied the one into the other, do produce any ratio. *As the ratio of A to C is compounded of the ratio's of A to B and B to C. For* $\frac{A}{B} + \frac{B}{C}$ *a* $= \frac{A}{C}$ *b* $= \frac{AB}{BC}$. a 10. *def.* 5. b 15. 4.

PROP. I.

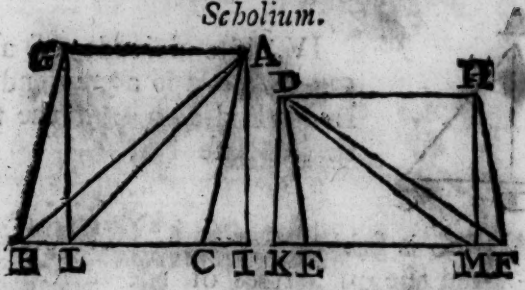


Triangles ABC, ACD, and parallelograms BCAE, CDFA which have the ſame height, are in proportion one to the other, as their baſes BC, CD are.

a Take as many as you pleaſe, BG, GH, equal to BC, and alſo DI = CD. and join AG, AH, AI. a 3. 1.

b The triangles ACB, ABG, AGH are equal, and *b* alſo the triangle ACD = ADI. Therefore the triangle ACH is as multiple of the triangle ACB, as the baſe HC is of the baſe BC; and the triangle ACI as multiple of the triangle ACD, as the baſe CI is of CD. But if HC ⊏, =, ⊐ CI, *c* then is likewiſe the triangle AHC ⊏, =, ⊐ ACI; and *d* therefore BC. CD :: the triangle ABC. ACD :: *e* Pgr. CE. CF. *Which was to be dem.* b 38. 1. c *ſch.* 38. 1. d 6. *def.* 5. e 41. 1. & 15. 5.

Scholium.

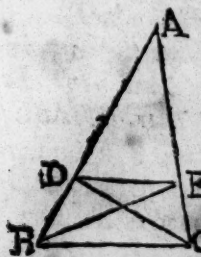


Hence, *Triangles ABC, DEF, and Pgrs. AGBC, DEFH, whose baſes BC, EF are equal, are in ſuch proportion as their altitudes, AI, DK are.*

a 3. 1. b 7. 5. c 1. 6. d 41. 1. & 15. 5.

a Take IL = CB, and KM = EF; and join LA, LG, MD, MH. then is it evident that the triangles ABC. DEF :: *b* ALI. DKM :: *c* AI. DK :: *d* pgrs. AGBC. DEFH. *Which was to be dem.*

PROP. II.



If to one ſide BC of a triangle ABC be drawn a parallel right line DE, the ſame ſhall cut the ſides of the triangle proportionally (AD. BD :: AE. EC.) And if the ſides of the triangle be proportionally cut (AD. BD :: AE. EC) then a right line DE joined at the ſections D, E, ſhall be parallel to the remaining ſide of the triangle BC. Draw CD and BE.

a 17. 1. b 7. 5. c 2. 6. d 11. 5. e 1. 6. f 19. 5. g 39. 1.

1. *Hyp.* Becauſe the triangle DEB *a* = DEC, *b* therefore ſhall be the triangle ADE. DBE : ADE, ECD. But the triangle ADE. DBE :: *c* AD. DB, and the triangle ADE. DEC :: AE. EC, *d* therefore AD. DB :: AE. EC.

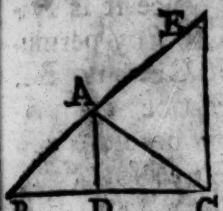
2. *Hyp.* Becauſe AD. DB :: AE. EC, *e* that is the triangle ADE DBE :: ADE. ECD; *f* therefore is the triangle DBE = ECD; and *g* therefore DE, BC are parallels. *Which was to be dem.*

Scholium.

If there be drawn many parallels to one ſide of any triangle, then all the ſegments of the ſides ſhall

ſhall be proportional; as is eaſily deducible from the precedent.

PROP. III.



If an angle BAC of a triangle BAC be biſected, and the right line AD, that biſects the angle, cut the baſe alſo; then ſhall the ſegments of the baſe have the ſame ratio that the other ſides of the triangle have (BD. DC :: AB. AC.) And if the ſegments of the baſe have the ſame ratio, that the other ſides of the triangle have (BD. DC :: AB. AC.) then a right line AD drawn from the top A to the ſection D, ſhall biſect that angle BAC of the triangle.

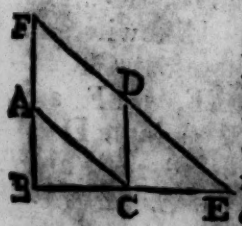
Produce BA, and make AE = AC, and join CE.

1. *Hyp.* Becauſe AE = AC, therefore is the angle ACE *a* = E *b* = ½ BAC *c* = DAC; *d* therefore DA, CE are parallels. *e* Wherefore BA. AE (AC) :: BD. DC.

2. *Hyp.* Becauſe BA. AC (AE) :: BD. DC, *f* therefore are DA, CE parallels; and *g* therefore is the angle BAD = E; and the angle DAC = ACE *h* = E. *k* therefore the angle BAD = DAC. Wherefore the angle BAC is biſected. *Which was to be dem.*

a 5. 1.
b 32. 1.
c *hyp.*
d 27. 1.
e 2. 6.
f 2. 6.
g 29. 1.
n 5. 1.
k 1. *ax.*

PROP. IV.



Of equiangular triangles ABC, DCE, the ſides are proportional which are about the equal angles, B, DCE, (AB. BC :: DC. CE, &c.) And the ſides AB, DC, &c. which are ſubtended under the equal angles ACB, E, &c. are homologous, or of like ratio.

Set the ſide BC in a direct line to the ſide CE, and produce BA and ED till they *a* meet.

a 32. 1.

b *hyp.* Because the angle B *b* = ECD, *c* therefore
c 28. 1. BF, CD are parallel; Also because the angle
BCA *b* = CED, *c* therefore are CA, EF parallel.
d 34. 1. Therefore the figure CAFD is a Pgr. *d* therefore
AF = CD, and AC = *d* ED. Whence it is evi-
e 2. 6. dent that AB. AF (CD) :: *e* BC. CE. *f* by permu-
f 16. 5. tation therefore AB. BC :: CD. CE. also BC.
CE :: FD (AC.) DE. *f* and thence by per-
g 22. 5. mutation BC. AC :: CE. DE. *g* Wherefore also
by equality AB. AC :: CD. DE. Therefore, *&c.*

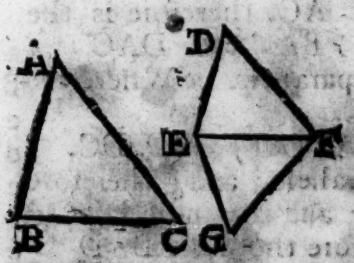
Coroll.

Hence AB. DC :: BC. CE :: AC. DE.

Schol.

Hence, If in a triangle FBE there be drawn AC a parallel to one side FE, the triangle ABC shall be like to the whole FBE.

PROP. V.

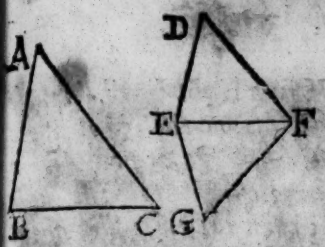


If two triangles ABC, DEF have their sides proportional (AB. BC :: DE. EF, and AC. BC :: DF, EF, and also AB. AC :: DE. DF) those triangles are equiangular, and those angles equal under which are subtended the homologous sides.

a 23. 1. At the side EF *a* make the angle FEG = B,
b 32. 1. and the angle EFG = C; *b* whence the angle
c 4. 6. G = A. Therefore GE. EF *c* :: AB. BC :: DE.
d *hyp.* EF. *e* and therefore GE = DE. Likewise GF.
e 11. 5. *and* EF *c* :: AC. CB :: *d* DF. FE. *e* therefore GF =
9. 5. DF. Therefore the triangles DEF, GEF are
f 8. 1. mutually equilateral. *f* Therefore the angle D =
g 32. 1. G = A, and the angle FED *f* = FEG = B, and *g* consequently the angle DEF = C. Therefore, *&c.*

PROP.

PROP. VI.

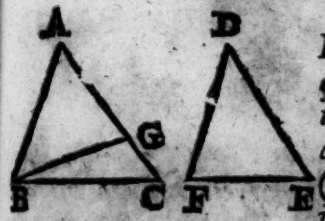


If two triangles ABC, DEF have one angle B equal to one angle DEF, and the sides about the equal angles B, DEF proportional (AB. BC :: DE. EF) then those triangles ABC, DEF are equiangular, and have those angles equal, under which are subtended the homologous sides.

At the side EF make the angle FEG = B, and the angle EFG = C; *a* then will the angle G = A. Therefore GE. EF :: *b* AB. BC :: *c* DE. EF, *d* and therefore DE = GE. But the angle DEF *e* = B *f* = GEF; therefore the angle D *g* = G = A, *h* and consequently the angle EFD = C. *W. W. to be dem.*

a 32. 1.
b 4. 6.
c *hyp.*
d 9. 5.
e *hyp.*
f *constr.*
g 4. 1.
h 32. 1.

PROP. VII.

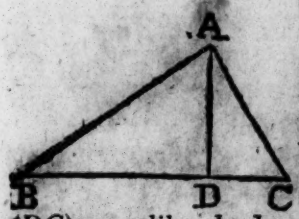


If two triangles ABC, DEF have one angle A equal to one angle D, and the sides about the other angles ABC, E, proportional (AB. BC :: DE. EF) and if they have both of the remaining angles C, F either less or not less than a right angle; then shall the triangles ABC, DEF be equiangular, and have those angles equal about which the proportional sides are.

For, if it can be, let the angle ABC ⌐ E, and make the angle ABG = E. Therefore, whereas the angle A *a* = D, *b* thence is the angle AGB = F. Therefore AB. BG *c* :: DE. EF :: *d* AB. BC. *e* therefore BG = BC. *f* therefore the angle BGC = BCG. *g* Therefore BGC or C is less than a right angle, and *h* consequently AGB or F is greater than a right: Therefore the angles C and F are not of the same species or kind, which is against the Hypothesis.

a *hyp.*
b 32. 1.
c 4. 6.
d *hyp.*
e 9. 5.
f 5. 1.
g *cor.* 17. 1.
h *cor.* 12. 1.

PROP. VIII.



If in a right-lined triangle ABC, from the right angle BAC there be drawn AD a perpendicular to the base BC; then the triangles about the perpendicular (ADB, ADC) are like both to the whole triangle ABC, and also one to the other.

a *hyp.*
b 17. *ax.*
c 32. & 4. 6
d 21. 6.
e 1. *def.* 8.

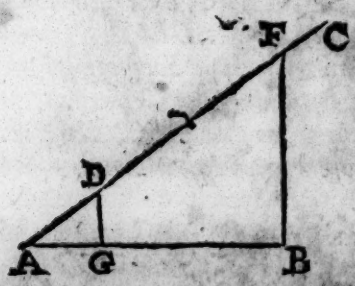
For because BAC, ADB are *a* right angles, *b* and so equal, and B common; the triangles BAC, ADB *c* are like. By the same argument BAC, ADC are like *d* whence also ADB, ADC will be like. *Which was to be dem.*

Coroll.

Hence, 1. BD. DA *e* :: DA. DC.

2. BC. AC :: AC. DC. and CB. BA :: BA. BD.

PROP. IX.



From a right line given AB to cut off any part required, as $\frac{1}{3}$ (AG.)

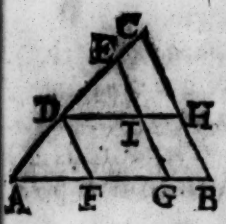
a 31. 1.
b 31. 1.
c 2. 6.
d 28. 5.

From the point A draw an infinite line AC any-wise, in which, *a* take any three equal parts AD, DE, EF. join FB. to which from D *b* draw the parallel DG. *And the thing is done.*

For GB. AG :: *c* FD. AD; whence by *d* composition AB. AG :: AF. AD. therefore, whereas AD = $\frac{1}{3}$ of AF, therefore is AG = $\frac{1}{3}$ of AB. *Which was to be done.*

PROP.

PROP. X.



To divide a right line given AB not divided (in F and G) *as another line given AC was cut* (in D and E.)

Let a right line BC join the extremities of the line divided, and of the line not divided; and to that line from the points E, D *a* draw the parallels EG, DF meeting with the right line that is to be cut, in G and F; then the thing is done.

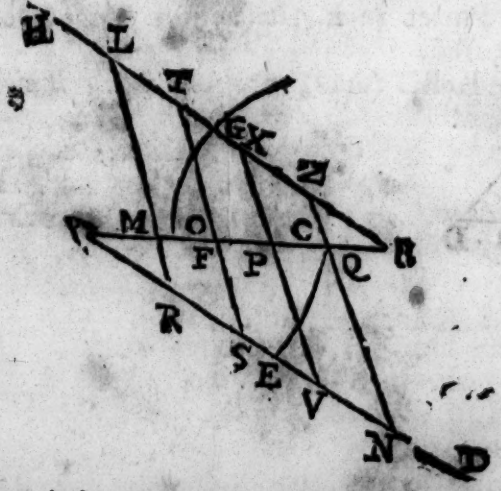
For let DH be *a* drawn parallel to AB. Then AB. DE :: *b* AF. FG. and DE. EC *b* :: DI. IH :: *c* FG. GB. *Which was to be done.*

a 31. 1.

b 2. 6.

c 34. 1. & 7. 5.

Schol.



Hence is learnt to cut a right line given AB into as many equal parts as you please (*suppose* 5;) which will be more easily performed thus.

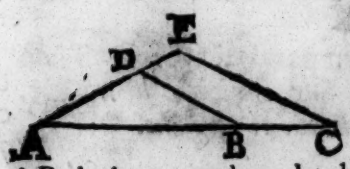
Draw an infinite line AD, and another BH parallel to it, and infinite also. Of those take equal parts, AR, RS, SV, VN; and BZ, ZX, XT, TL;

in

in each line leſs parts by one, than are required in AB; then let the right lines LR, TS, XV, ZN be drawn; theſe lines ſo drawn ſhall cut the right line given AB into five parts.

a 33. 1. b *conſtr.* c 2. 6. For RL, ST, VX, NZ are *a* parallels; therefore, whereas AR, RS, SV, VN are *b* equal; *c* thence AM, MO, OP, PQ are equal alſo. Likewiſe, becauſe that BZ = ZX, therefore is BQ = PQ. and therefore AB is cut into five parts. *Which was to be dem.*

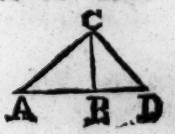
PROP. XI.



Two right lines being given AB, AD, to find out a third in proportion to them (DE.)

Join BD, and from AB being produced take BC = AD. Thro' C draw CE parallel to DB; with which let AD produced meet in E. then is DE the proportional required.

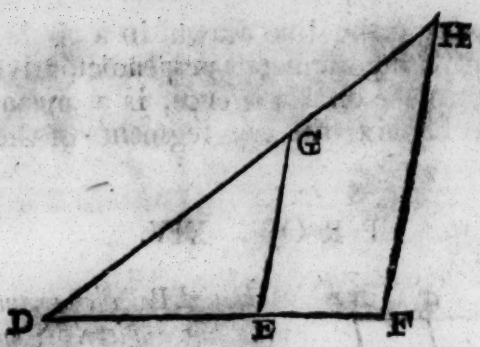
a 2. 6. For AB. BC (AD) *a* :: AD. DE. *Which was to be dem.*



b *cor.* 9. Or thus: Make the angle ABC right, and alſo the angle ACD right, then *b* AB. BC :: BC. BD.

PROP.

PROP. XII.

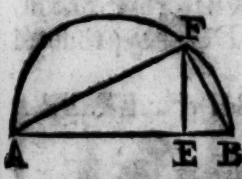


Three right lines being given DE, EF, DG to find out a fourth proportional GH.

Join EG, and thro' F draw FH parallel to EG; with which let DG produced to H meet. Then it is evident that DE. EF *a* :: DG. GH. *Which was to be done.*

a 2. 6.

PROP. XIII.



Two right lines being given AE, EB, to find out a mean proportional EF.

Upon the whole line AB as a diameter deſcribe a ſemicircle AFB, and from E erect a perpendicular EF meeting with the periphery in F. then AE. EF :: EF. FB. For let AF and FB be drawn; *a* then from the right angle of the right-angled triangle AFB is drawn a right line FE perpendicular to the baſe. *b* Therefore AE. FE :: FE. EB. *Which was to be done.*

a 31. 1.

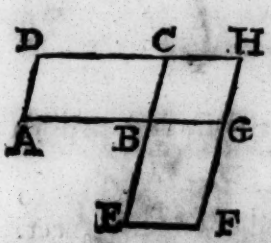
b *cor.* 8. 6.

Coroll.

Coroll.

Hence, A right line drawn in a circle from any point of the diameter perpendicularly, and extended to the circumference, is a mean proportional betwixt the two segments of the diameter.

PROP. XIV.



Equal Parallelograms having one angle ABC equal to one EBG, have the sides BD, BF which are about the equal angles reciprocal (AB. BG :: EB. BC;) And those Parallelograms BD, BF which have one angle ABC equal to one EBG, and the sides which are about the equal angles reciprocal, are equal.

For let the sides AB, BG about the equal angles make one right line; *a* wherefore EB, BC shall do the same. Let FG, DC be produced till they meet.

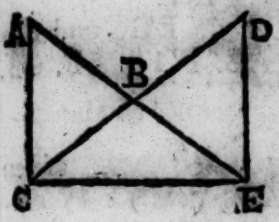
1. *Hyp.* AB. BG *b* :: *f* BD. BH :: *c* BF. BH :: *d* BE. BC. *e* therefore, *&c.*

2. *Hyp.* BD. BH :: *f* AB. BG :: *g* BE. BC :: *h* BF. BH. *k* Therefore the Pgr. BD = BF. *Which was to be dem.*

a *sch.* 15. 1.
b 1. 6.
c 7. 5.
d 1. 6.
e 11. 5.
f 1. 6.
g *hyp.*
h 1. 6.
k 11. *and* 9. 5.

PROP.

PROP. XV.



Equal triangles having one angle ABC equal to one DBE, their ſides which are about the equal angles are reciprocal (AB. BE :: DB. BC.) And thoſe triangles that have one angle ABC equal to one DBE, and have alſo the ſides that are about the equal angles reciprocal (AB. BE :: DB. BC) are equal.

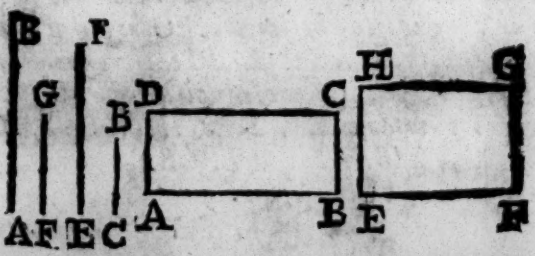
Let the ſides CB, BC, which are about the equal angles be ſet in a ſtrait line; *a* therefore ABE is a right line. Let CE be drawn.

1. *Hyp.* AB.BE :: *b* the triangle ABC.CBE *c* :: the triangle DBE.CBE :: *d* DB.BC. *e* therefore, &c.

1. *Hyp.* The triangle ABC. CBE :: *f* AB. BE :: *g* DB. BC *h* :: the triangle DBC. CBE. *k* Therefore the triangle ABC = DBE. *Which was to be dem.*

a *ſch.* 15. 1.
b 1. 6.
c 7. 5.
d 1. 6.
e 11. 5.
f 1. 6.
g *hyp.*
h 1. 6.
k 11, *and* 9. 5.

PROP. XVI.



If four right lines be proportional (AB. FG :: EF. CB) the rectangle AC comprehended under the extremes AD, CB, is equal to the rectangle EG comprehended under the means FG, EF. And if the rectangle AC comprehended under the extremes AB, CB be equal to the rectangle EG comprehended under the means FG, EF, then are the four right lines proportional. (AB. FG :: EF. CB.)

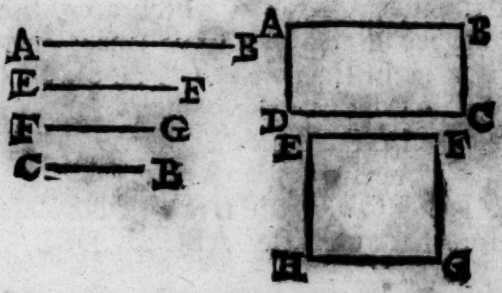
1. *Hyp.*

a 12. *ax.* 1. *Hyp.* The angles B and F are right, and *a* consequently equal, and by Hyp. AB. FG :: EF. CB.
b 14. 6. *b* therefore the rectangle AC = EG.
c *hyp.* 2. *Hyp.* The rectangle AC *c* = EG, and the
d 14. 6. angle B = F; *d* therefore AB. FG :: EF. CB.
Which was to be dem.

Coroll.

Hence, it is easy to apply a rectangle given
e 12. 6. EG to a right line given AB; (*viz.*) *e* by making AB. EF :: FG. BC.

PROP. XVII.



If three right lines be proportional (AB. EF :: EF. CB.) the rectangle AC made under the extremes AB, CB is equal to the square EG made of the middle EF. And if the rectangle AC comprehended under the extremes AB, CB be equal to the square EG made of the middle EF, then the three lines are proportional, (AB. EF :: EE. CB.)

Take FG = EF.

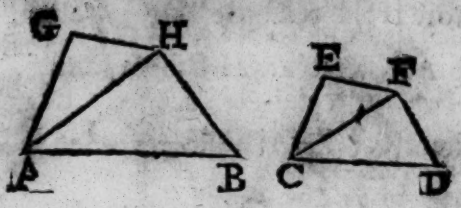
a *hyp.* 1. *Hyp.* AB. EF :: *a* EF (FG.) CB. therefore
b 16. 6. the rectangle AC *b* = EG *c* = EFq.
c 29. *def.* 1. 2. *Hyp.* The rectangle AC *d* = to the square
d *hyp.* EG = EFq. *e* therefore AB. EF :: FG (EF.) BC.
e 16. 6. *Which was to be dem.*

Coroll.

Let A × B = Cq. therefore A. C :: C. B.

PROP.

PROP. XVIII.

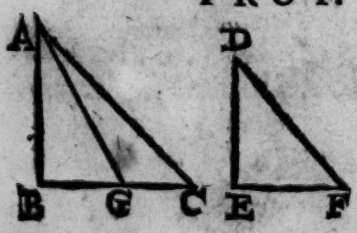


From a right line given AB to deſcribe a right-lined figure AGHB like and alike ſituate to a right-lined figure given CEFD.

Reſolve the right-lined figure given into triangles; *a* Make the angle ABH = D, *a* and the angle BAH = DCF, *a* and the angle AHG = CFE, *b* and the angle HAG = FCE. then AGHB ſhall be the right-lined figure ſought. a 23. 1.

For the angle B *b* = D, and the angle BAH *b* = DCF. *c* wherefore the angle AHB = CFD. *b* alſo the angle HAG = FCE, and the angle AHG *b* = CFE. *c* wherefore the angle G = E, and the whole angle GAB *d* = ECD, and the whole angle GHB *d* = EFD. The Polygones therefore are mutually equiangular. Moreover becauſe the triangles are equiangular, therefore AB. BH *e* :: CD. DF; and AG. GH *e* :: CE. EF. Likewiſe AG. AH :: *e* CE. CF. and AH. AB :: CF. CD. *f* From whence by equality AG. AB :: CF. CD. After the ſame manner GH. HB :: EF. FD. Therefore the Polygones ABHG, CDEF are like and alike ſituate. *Which was to be done.* b *conſtr.* c 32. 1. d 2. *ax.* e 4. 6. f 22. 5. g 6. *def.* 5.

PROP. XIX.



Like triangles ABC, DEF are in duplicate ratio of their homologous ſides, BC, EF.

a Let there be made BC, EF :: EF. BG. and let AG be drawn. Becauſe that AB. a 11. 6.

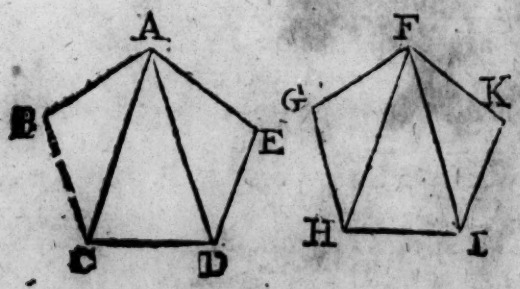
DE

b *cor.* 4. 6. DE *b* : BC. EF *c* :: EF. BG, and the angle B = E.
c *constr.* *d* therefore is the triangle ABG = DEF. But the
d 15. 6. triangle ABC. ABG :: *e* BC. BG, and *f* $\frac{BC}{BG} = \frac{BC}{EF}$
e 1. 6.
f 10. *def.* 5. twice; therefore $\frac{ABC}{ABG}$ that is $\frac{ABC}{DEF}$ *g* $= \frac{BC}{EF}$ twice.
g 11. 5. *Which was to be dem.*

Coroll.

Hence, If three right lines (BC, EF, BG) be proportional, then as the first is to the third, so is a triangle made upon the first BC, to a triangle like and alike described upon the second EF: or so is a triangle described upon the second EF to a triangle like and alike described upon the third.

PROP. XX.



Like Polygones ABCDE, FGHIK are divided into equal triangles ABC, FGH, and ACD, FHI, and ADE, FIK; both equal in number, and homologous to the wholes (ABC. FGH :: ABCDE. FGHIK :: ACD. FHI :: ADE. FIK.) And the Polygones ABCDE, FGHIK have a double ratio one to the other what one homologous side BC has to the other homologous side GH.

a *hyp.* 1. For the angle B *a* = G, and AB. BC *a* :: FG.
b 6. 6. GH. *b* therefore the triangles ABC, FGH are

are equiangular. After the same manner are the triangles AED, FKI like. Whereas therefore the angle BCA *b* = GHF, and the angle ADE *b* = FIK, and the whole angles BCD, GHI, and the whole angles CDE, HIK are *c* equal, there remains the angle ACD *d* = FHI, and the angle ADC = FIH. *e* from whence also the angle CAD = HFI. therefore the triangles ACD, FHI are like. Therefore, *&c.*

b 6. 6.
c *hyp.*
d 3. *ax.*
e 32. 1.

2. Because that the triangles BCA, GHF are like, *f* therefore is $\frac{BCA}{GHF} = \frac{BC}{GH}$ twice. For the same reason is $\frac{CAD}{HFI} = \frac{CD}{HI}$ twice; lastly $\frac{DEA}{IKF} = \frac{DE}{IK}$ twice. now whereas that BC. GH *g* :: CD. HI *g* :: DE. IK. *h* therefore is the triangle BCA. GHF :: CAD. HFI :: DEA. IKF *k* :: the polygone ABCDE. FGHIK :: $\frac{BC}{GH}$ twice.

f 19. 6.
g *hyp. and* 16. 5.
h *cor.* 23. 5.
k 12. 5.

Coroll.

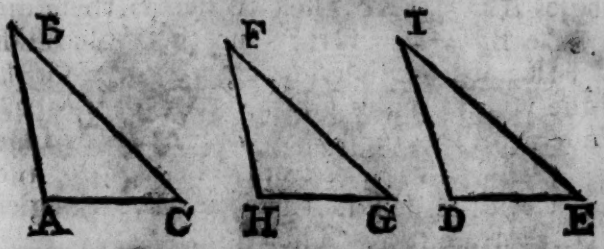
1. Hence, If there be three right lines proportional, then as the first is to the third, so is a polygone made upon the first to a polygone made on the second, like and alike described; or so is a polygone described upon the second to a polygone made on the third like and alike described.

By which is found out a method of enlarging or diminishing any right-lined figure in a ratio given: As if you would make a pentagone quintuple of that pentagone whereof CD is the side, then betwixt AB and 5 AB find out a mean proportional * upon this raise a pentagone like to that given, and it shall be quintuple of the pentagone given.

* 18. 6.

2. Hence also, If the homologous sides of like figures be known, then will the proportion of the figures be evident, *viz.* by finding out a third proportional.

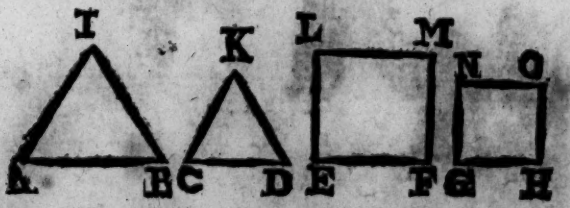
PROP. XXI.



Right lined figures ABC, DIE which are like to the same right-lined figure HFG, are also like one to the other.

a 1. *def.* 6. For the angle A *a*=H *a*=D; and the angle C *a*=G *a*=E; and the angle B *a*=F *a*=I. Also *a* AB.AC :: HF.HG :: DI.DE. & *a* AC.CB :: HG.GF :: DE.EI. And AB.BC :: HF.FG :: DI.IE. Therefore ABC, DIE are alike. *Which was to be dem.*

PROP. XXII.



If four right lines be proportional (AB. CD :: EF. GH) the right-lined figures also described upon them being like and in like sort situate, shall be proportional (ABI. CDK :: EM. GO.) And if the right-lined figures described upon the lines, like and alike situate, be proportional (ABI. CDK :: EM. GO.) then the right lines also shall be proportional (AB. CD :: EF, GH.)

a 19. 6. 1. *Hyp.* $\frac{ABI}{CDK} a= \frac{AB}{CD}$ twice $a= \frac{EF}{GH}$ twice $a= \frac{EM.}{GO}$

b *hyp.* *b* therefore ABI. CDK :: EM. GO.

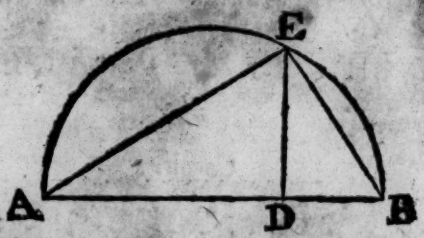
c 20. 6. d *cor.* 23. 5. 2. *Hyp.* $\frac{AB}{CD}$ twice $a= \frac{ABI}{CDK} b= \frac{EM}{GO} =c \frac{EF}{GH}$ twice. Therefore AB. CD :: EF. GH. *Which was to be dem.*

Schol.

Schol.

Hence is deduced the manner and reaſon of multiplying ſurd quantities. ex. gr. Let $\sqrt{5}$ to be multiplied into $\sqrt{3}$. I ſay that the product will be $\sqrt{15}$. For by the definition of multiplication it ought to be, as 1. $\sqrt{3}$ :: $\sqrt{5}$. to the product. Therefore by this q. 1. q. $\sqrt{3}$ :: q. $\sqrt{5}$. q. of the product. therefore the ſquare of the product is 15. Wherefore $\sqrt{15}$ is the product of $\sqrt{3}$ into $\sqrt{5}$. *Which was to be dem.*

THEOREM.



If a right line AB be cut any-wiſe in D, the rectangle comprehended under the parts AD, DB is a mean proportional betwixt their ſquares. Likewiſe the rectangle comprehended under the whole AB and one part AD, or DB, is a mean proportional betwixt the ſquare of the whole AB and the ſquare of the ſaid part AD, or DB. Pet. Herig.

Upon the diameter AB deſcribe a ſemicircle; from D erect a perpendicular DE meeting with the periphery in E. join AE. BE.

It's evident that AD. DE *a* :: DE. DB. *b* therefore ADq DEq :: DEq. DBq. *c* that is, ADq. ADB :: ADB. DBq. *Which was to be dem.*

a *cor.* 8. 6.
b 22. 6.
c 17. 6.

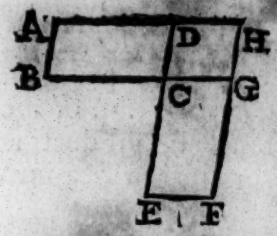
Moreover BA. AE :: *d* AE. AD. *e* therefore BAq. AEq :: AEq. ADq. *f* that is, BAq. BAD :: BAD. ADq. After the ſame manner ABq. ABD :: ABD. BDq. *Which was to be dem.*

d *cor.* 8. 6.
e 22. 6.
f 17. 6.

Or thus: ſuppoſe Z = A + E. It is manifeſt that Aq. AE :: *a* A. E :: *a* AE. Eq. alſo Zq. ZA :: *a* Z. A :: ZA. Aq. and Zq. ZE :: *a* Z. E :: ZE. Eq.

a 1. 6.

PROP. XXIII.



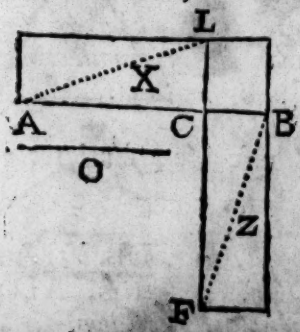
Equiangular parallelograms AC, CF, have the ratio one to the other, which is compounded of their sides. $\left(\frac{AC}{CF} = \frac{BC}{CG} + \frac{DC}{CF}\right)$

Let the sides about the equal angles C be *a* set in a direct line, and let the Pgr. CH be completed. Then is the ratio of $\frac{AC}{CF}$ $^{b} = \frac{AC}{CH} + \frac{CH}{CF}$ $^{c} = \frac{BC}{CG} + \frac{DC}{CE}$. *Which was to be dem.*

a *sch.* 15.
b 20. *def.* 5.
c 1. 6.

Coroll.

Andr. Tacq. 15. 5. Hence, *and from* 34. 1. *it appears,* 1. *That triangles which have one angle equal (as C) have a ratio compounded of the ratio's of the right lines (AC to CB, and LC to CF) containing the equal angle.*



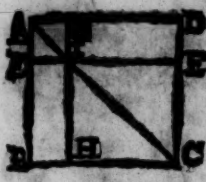
* 35. 1. 2. *That all rectangles, and* * *consequently all parallelograms, have their ratio one to the other compounded of the ratio's of base to base, and altitude to altitude.* After the like manner you may argue in triangles.

3. *From hence is apparent how to give the proportion of triangles and parallelograms.* Let there be two Pgrs. X and Z, whose bases are AC, CB, and altitudes CL, CF. Make CL. CF :: CB. O. * then will it be X. Z :: AC. O.

* 14. 6. *and* 1. 6.

PROP.

PROP. XXIV.

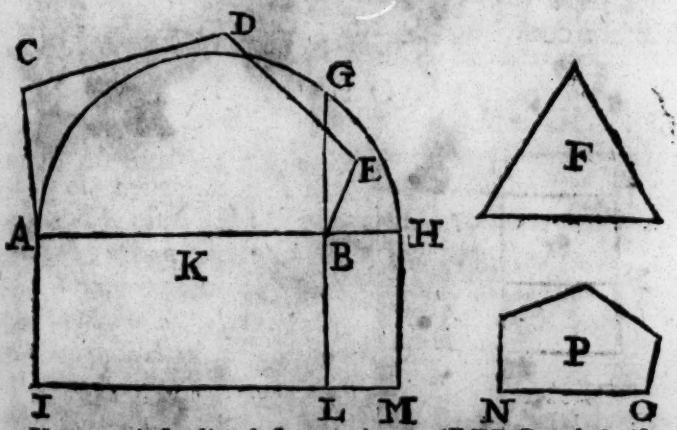


In every parallelogram ABCD, the parallelograms EG, HF which are about the diameter AC are like to the whole, and also one to the other.

For the Pgrs. EG, HF have each of them one angle common with the whole, *a* therefore they are equiangular to the whole, and also one to the other. Also both the triangles ABC, AEI, IHC *a* and the triangles ADC, AGI, IFC are equiangular mutually; *b* therefore AE. EI :: AB. BC, and *b* AE. AI :: AB. AC, and *b* AI. AG :: AC. AD. *c* Therefore by equality, AE. AG :: AB. AD. *d* Therefore the Pgrs. EG, BD are like. After the same manner are HF, BD like also. Therefore, *&c.*

a 29. 1.
b 4. 6.
c 22. 5.
d 1. *def.* & 6.

PROP. XXV.



Unto a right-lined figure given ABEDC to describe another figure P like and alike situate, which also shall be equal to another right-lined figure given F.

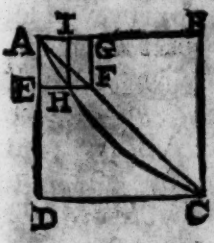
a Make the rectangle AL = ABEDC; *b* also upon BL make the rectangle BM = F; betwixt AB and BH *c* find out a mean proportional NO; Upon NO *b* make the polygone B like to the right-lined figure given ABEDC. I say the polygone

a 45. 1.
b 44. 1.
c 13. 6.
d 18. 6.

e *cor.* 20. 6. f 1. 6. lygone P so made shall be equal to F that was given.

g 14. 5. h *constr.* For ABEDC (AL) P :: *e* AB. BH :: *f* AL. BM. Therefore P *g* = BM. *b* = F. *Which was to be done.*

PROP. XXVI.

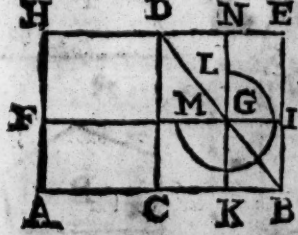


If from a Parallelogram ABCD, be taken away another Parallelogram AGFE, like unto the whole, and in like sort set, having also an angle common with it EAG; then is that Parallelogram about the same diagonal AC with the whole.

If you deny AC to be the common diagonal, then let AHC be it, cutting EF in H, and let HI be drawn parallel to AE. Then are Pgrs. EI, DB *a* like. *b* therefore AE. EH :: AD. DC :: *c* AE. EF. and *d* consequently EH = EF. *f Which is absurd.*

a 24. 6. b 1. *def.* 6. c *hyp.* d 9 5. f 9. *ax.*

PROP. XXVII.



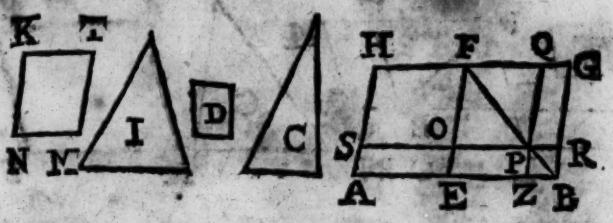
Of all parallelograms, AD, AG applied to the same right line AB, and wanting in figure by the parallelograms CE, KI like and alike set to the Pgr. AD which is described upon the half line, the greatest is that AD which is applied to the half being like to the defect KI.

For because that GE *a* = GC, and KI added in common. *b* thence is KE = CI *c* = AM. add CG in common. *d* then is AG = to the Gnomon MBL. But the Gnomon MBL *e* ⌐ CE (AD.) Therefore AG ⌐ AD. *Which was to be demonstrated.*

a 43. 1. b 2. *ax.* c 36. 1. d 2. *ax.* e 9. *ax.*

PROP.

PROP. XXVIII.



Upon a right line given AB, to apply a parallelogram AP equal to a right-lined figure given C, deficient by a parallelogram ZR which is like to another parallelogram given D. * *Now it is requiſite that the right-lined figure given C, whereunto the Pgr. to be applied AP muſt be equal, be not greater than the Pgr. AF which is applied upon the half line, the defects being like, namely the defect of the Pgr. AF, which is applied to the half line, and the defect of the Pgr. D to be applied whoſe defect is to be like to the Pgr. given.* * 27. 6.

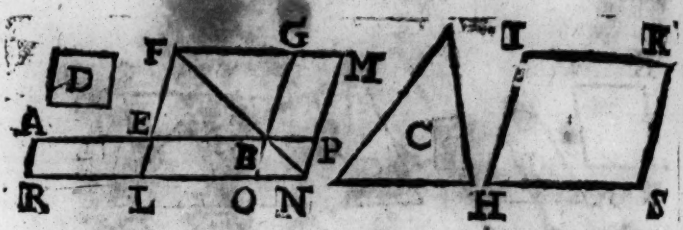
Biſect AB in E; upon EB *a* make the Pgr. EG like to the Pgr. D; and *b* let EG = C + I. *c* Make the Pgr. NT = I, and like to the Pgr. given D, or EG; Draw the diameter FB; Make FO = KN, and FQ = KT; thro' O and Q draw the parallels SR, QZ. Then is the Pgr. AP that which was ſought.

a 18. 6.
b *ſch.* 45. 1.
c 25. 6.

For the Pgrs. D, EG, OQ, NT, ZR are all like one to the other, and the Pgr. EG = *e* NT + C = *e* OQ + C. *f* wherefore C = to the Gnomon OBQ *g* = AO + PG = *h* AO + EP = AP, *Which was to be done.*

d *conſtr.* &
24. 6.
e *conſtr.*
f 3. *ax.*
g 2. *ax.*
h 43. 1.

PROP. XXIX.

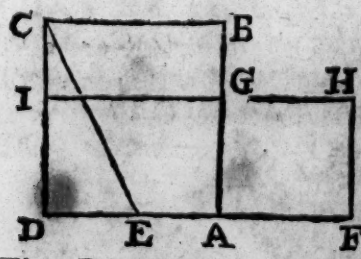


Upon a right line given AB. to apply a Pgr. AN equal to a right-lined figure given C, exceeding by a Pgr. OP, which shall be like to another Pgr. given D.

a 18. 6. b 25. 6. c 3. 1. Biſect AB in E. Upon EB *a* make a Pgr. EG like to D which was given. And *b* let the Pgr. HK = EG + C, and like to D given, or EG. Make FEL = *c* IH, and *c* FGM = IK. Thro' L, M draw the parallels MN and RN, and AR parallel to NM. produce ABP, GBO. Draw the diameter FBN. Then is AN the parallelogram required.

d *conſtr.* e 24. 1. f *conſtr.* g 3. *ax.* h 36. 1. k 43. 1. l 2. *and* 1. *ax.* For the pgrs. D, HK. LM. EG are *d* like, *e* therefore the pgr. OP is like to the pgr. LM, or D. Alſo LM *f* = HK *f* = EG + C. *g* Therefore C = to the Gnomon ENG. But AL *h* = LB *k* = BM. *l* therefore C = AN. *Which was to be done.*

PROP. XXX.

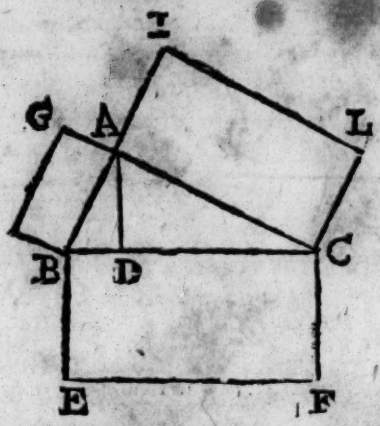


To cut a finite right line given AB according to extreme and mean ratio (AB. AG :: AG. GB.)

a 11. 2. b 17. 6. *a* Cut AB in G, in ſuch wiſe that AB × BG = AGq. *b* Then BA. AG :: AG. GB. *Which was to be done.*

PROP.

PROP. XXXI.



In right-angled triangles BAC, any figure BF described upon the side BC subtending the right angle BAC, is equal to the figures BG, AL, described upon the sides BA, AC, containing the right angle, like and alike situate to the former BF.

From the right angle BAC let down a perpendicular AD. Because that DC. CA :: *a* CA. CB, *b* therefore AL. BF :: DC. CB. Also, because DB. BA :: *a* BA. BC, *b* therefore BG. BF :: DB. BC. *c* therefore AL + BG. BF :: DC + DB (BC.) BC. *d* Therefore AL + BG = BF. *Which was to be dem.*

a *cor.* 8. 6.
b *cor.* 20. 6.
c 24. 5.
d *schol.* 14. 5.

Or thus: BG. BF :: *e* BAq. BCq. And *e* AL. BF :: ACq. BCq. *f* therefore BG + AL. BF :: BAq + ACq. BCq. *g* Therefore whereas BAq + ACq = *b* BCq. *h* thence is BG + AL = BF. *Which was to be dem.*

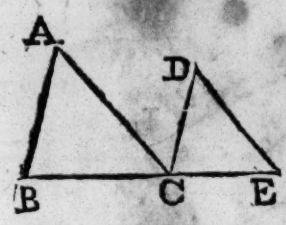
e 22. 6.
f 24. 5.
g *sch.* 14. 5.
h 47. 1.

Coroll.

From this Proposition you may learn how to add or substract any like figures, by the same method that is used in adding and substracting of squares, in *Schol.* 47. 1.

PROP.

PROP. XXXII.

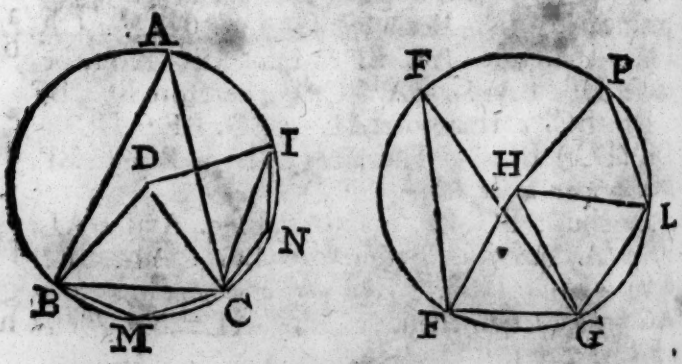


If two triangles ABC, DCE having two sides proportional to two (AB. AC :: DC. DE.) be so compounded or set together at one angle ACD that their homologous sides be also parallel AB to DC, and AC to DE,) then the remaining sides of those triangles (BC, CE) shall be found placed in one strait line.

a 29. 2. b *hyp.* c 6. 6. d 2. *ax.* e 32. 1. f 1. *ax.* g 14. 1.

For the angle A *a* = ACD *a* = D, and AB. AC *b* :: DC. DE. *c* therefore the angle B = DCE. Therefore the angle B + A *d* = ACE. But the angle B+A+ACB *e* = 2 right. *f* therefore the angle ACE + ACB = 2 right. *g* therefore BCE is a right line. *Which was to be dem.*

PROP. XXXIII.



In equal circles DBCA, HFGP, the angles BDC, FHG have the same ratio with their peripheries BC, FG on which they insist; whether the angles be set at the centers (as BDC, FHG) *or at the circumferences, A, E: And in like sort are the sectors BDC, FHG, because described upon the centers.*

Draw

Draw the right lines BC, FG. Make CI=CB, and GL=FG=LP, and join DI, HL, HP.

The arch BC *a*=CI. *a* alſo the arch FG, GL, LP are equal. *b* therefore the angle BDC=CDI, *b* and the angle FGH=GHL=LHP. Therefore the arch BI is as multiple of the arch BC, as the angle BDI is of the angle BDC. And in like manner is the arch EP as multiple of the arch FG, as the angle FHP is of the angle FHG. But if the arch BI ⊏, =, ⊐ FP. *c* then likewiſe is the angle BDI ⊏, =, ⊐ FHP. Therefore is the arch BC. FG :: *d* the angle BDC. FHG *e* :: $\frac{BDC}{2}$. $\frac{FHG}{2}$ *f* :: A. E. *Which was to be demonſtrated.*

a 28. 3.
b 27. 3.
c 27. 3.
d 6. *def.* 5.
e 15. 5.
f 20. 3.

Moreover, the angle BMC *g* = CNI; *h* and therefore the ſegment BCM = CIN. *k* Alſo the triangle BDC = CDI, *l* wherefore the ſector BDCM = CDIN. After the ſame manner are the ſectors FHG, GHL, LHP equal one to the other. Therefore ſince accordingly as the arch BI ⊏, =, ⊐ FGP, ſo is likewiſe the ſector BDI ⊏, =, ⊐ FHP; *m* thence ſhall be the ſector BDC. FHG :: the arch BC. FG. *Which was to be demonſtrated.*

g 27. 3.
h 24. 3.
k 4. 1.
l 2. *ax.*
m 6. *def.* 5

Coroll.

1. Hence, *As ſector is to ſector, ſo is angle to angle.* 11. 5.

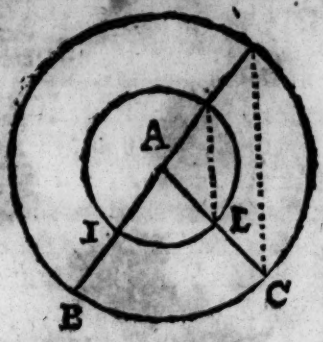
2. *The angle BDC in the center is to four right angles, as the arch BC, on which it inſiſts, to the whole circumference.*

For as the angle BDC is to a right angle, ſo is the arch BC to a quadrant. Therefore BDC is ſo to four right angles as the arch BC is to four quadrants, that is, the whole circumference. Alſo the angle A. 2 right :: the arch BC periphery.

3. Hence, *The arches IL, BC of unequal circles which ſubtend equal angles, whether at the centers, as IAL and BAC, or at the periphery are like.*

For

For IL. periph. :: angle IAL (BAC) 4 right. Also Arch BC. periph :: angle BAC. 4 right. Therefore IL. periph :: BC. periph. And consequently the arches IL and BC are like. Whence



4. *Two ſemidiameters AB, AC cut off like arches IL, BC from concentrical peripheries.*

The End of the ſixth Book.

THE

THE SEVENTH BOOK OF EUCLIDE's ELEMENTS.

Definitions.

I. UNity is that, by which every thing that is, is called One.

II. Number is a multitude compoſed of units.

III. One number is a part of another, the leſſer of the greater, when the leſſer meaſures the greater.

Every part is denominated from that number by which it meaſures the number whereof it is a part; as 4 *is called the third part of* 12, *becauſe it meaſures* 12 *by* 3.

IV. But the leſſer number is termed Parts, when it meaſures not the greater.

All parts whatſoever are denominated from thoſe two numbers, by which the greateſt common meaſure of the two numbers meaſures each of them; as 10 *is ſaid to be* ⅔ *of the number* 15; *becauſe the greateſt common meaſure, which is* 5, *meaſures* 10 *by* 2, *and* 15 *by* 3.

V. A number is Multiple (or Manifold) a greater in compariſon of a leſſer, when the leſſer meaſures the greater.

VI. An even number is that which may be divided into two equal parts.

VII. But an odd number is that which cannot be divided into two equal parts; or, that which differs from an even number by an unit.

VIII. A number evenly even is that which an even number meaſures by an even number.

IX. But a number evenly odd is that which an even number meaſures by an odd number.

X. A

X. A number oddly odd is that which an odd number meaſures by an odd number.

XI. A prime (or firſt) number is that, which is meaſured only by an unit.

XII. Numbers prime the one to the other, are ſuch as only an unit doth meaſure, being their common meaſure.

XIII. A compoſed number is that which ſome certain number meaſures.

XIV. Numbers compoſed the one to the other, are they, which ſome number, being a common meaſure to them both, does meaſure.

In this, and the preceding definition, unity is not a number.

XV. One number is ſaid to multiply another, when the number multiplied is ſo often added to it ſelf, as there are units in the number multiplying, and another number is produced.

Hence in every multiplication a unit is to the multiplier, as the multiplicand is to the product.

Obſ. *That many times, when any number are to be multiplied as (A into B) the conjunction of the letters denotes the product: So* $AB = A \times B$, *and* $CDE = C \times D \times E$.

XVI. When two numbers multiplying themſelves produce another, the number produced is called a plain number; and the numbers which multiplied one another, are called the ſides of that number: *So* 2 (C) × 3 (D) = 6 = CD *is a plain number.*

XVII. But when three numbers multiplying one another produce any number, the number produced is termed a ſolid number; and the numbers multiplying one another, are the ſides thereof: *So* 2 (C) × 3 (D) × 5 (E) = 30 = *CDE is a ſolid number.*

XVIII. A ſquare number is that which is equally equal; or, which is contained under two equal numbers. *Let A be the ſide of a ſquare; the ſquare is thus noted, AA, or Aq. or* A^2.

XIX. A

XIX. A Cube is that number which is equally equal equally, or, which is contained under three equal numbers. *Let A be the ſide of a Cube; the Cube is thus noted, AAA, or Ac. or A³.*

In this definition, and the three foregoing, unity is a number.

XX. Numbers are proportional, when the firſt is as multiple of the ſecond, as the third is of the fourth; or, the ſame part; or, when a part of the firſt number meaſures the ſecond, and the ſame part of the third meaſures the fourth, equally: and on the contrary. *So A. B :: C. D, that is,* 3. 9 :: 5. 15.

XXI. Like plane, and ſolid numbers are they, which have their ſides proportional: *Namely, not all the ſides, but ſome.*

XXII. A perfect number is that which is equal to all its aliquot parts.

As 6 *and* 28. *But a number that is leſs than its aliquotparts is called an* Abounding *number; and a greater a* Diminutive: *ſo* 12 *is an Abounding,* 15 *a Diminutive number.*

XXIII. One number is ſaid to meaſure another, by that number, which, when it multiplies, or is multiplied by it, it produces.

In Diviſion, a unit is to the quotient as the diviſor is to the dividend. Note, *that a number placed under another with a line betwixt them, ſignifies Diviſion:* So $\frac{A}{B}$ = A divided by B, and $\frac{CA}{B}$ = C × A divided by B.

Two numbers are called Terms or Roots of Proportion, leſſer than which cannot be found in the ſame proportion.

Poſtulates or Petitions.

1. THat numbers equal or manifold to any number may be taken at pleaſure.

2. That a greater number may be taken than any number whatſoever.

3. That

3. That Addition, Subſtraction, Multiplication, Diviſion, and the Extractions of roots or ſides of ſquare and cube numbers, be alſo granted as poſſible.

Axiomes.

1. WHatſoever agrees with one or many equal numbers, agrees likewiſe with the reſt.

2. Thoſe parts that are the ſame to the ſame part, or parts, are the ſame amongſt themſelves.

3. Numbers that are the ſame parts of equal numbers, or of the ſame number, are equal amongſt themſelves.

4. Thoſe numbers, of whom the ſame number, or equal numbers, are the ſame parts, are equal amongſt themſelves.

5. Unity meaſures every number by the units that are in it; that is, by the ſame number.

6. Every number meaſures it ſelf by a unity.

7. If one number, multiplying another, produce a third, the multiplier ſhall meaſure the product by the multiplicand; and the multiplicand ſhall meaſure the ſame by the multiplier.

Hence, *No prime number is either a plane, ſolid, ſquare, or cube number.*

8. If one number meaſures another, that number by which it meaſures ſhall meaſure the ſame by the units that are in the number meaſuring, that is by the number itſelf that meaſures.

9. If a number meaſuring another, multiply that by which it meaſures, or be multiplied by it, it produces that number which it meaſures.

10. How many numbers ſoever any number meaſures, it likewiſe meaſures the number compoſed of them.

11. If a number meaſure any number, it alſo meaſures every number which the ſaid number meaſures.

12. A number that meaſures the whole and a part taken away doth alſo meaſure the reſidue.

PROP.

PROP. I.

A....E..G.B 8 5 3
C..F..D 5/3 3/2 2/1
H---

If two unequal numbers AB, CD, being given, the leſſer CD be continually taken from the greater AB (and the reſidue EB from CD, &c.) by an alternate ſubſtraction, and the number remaining do never meaſure the precedent, till the unity GB be taken; then are the numbers which were given AB, CD, prime the one to the other.

If you deny it, let AB, CD have a common meaſure, namely, the number H. Therefore H meaſuring CD, does *a* alſo meaſure AE; and a 11. *ax.* 7.
conſequently the remainder EB; *a* therefore it likewiſe meaſures CF, and *b* ſo the remainder b 12. *ax.* 7.
FD; *a* wherefore it alſo meaſures EG. But it meaſured the whole EB, and *b* therefore it muſt meaſure that which remains GB, a unity, itſelf being a number. *c Which is abſurd.* c 9. *ax.* 7.

PROP. II.

9 6
A.....E.....B 15 9 6
6 3
C.....F...D 2/6 1/3 3/0
G---

Two numbers AB, CD being given, not prime the one to the other to find out their greateſt common meaſure FD.

Take the leſſer number CD from the greater AB as often as you can. If nothing remains, *a* a 6. *ax.* 7.
it is manifeſt that CD is the greateſt common meaſure. But if there remains ſomething (as EB) then take it out of CD, and the reſidue FD out of EB, and ſo forward till ſome number (FD) meaſure the ſaid EB (*b* for this will be before you come to a unity.) FD ſhall be the greateſt common meaſure. b 1. 7.

For FD *c* meaſures EB, and *d* therefore alſo CF; c *conſtr.*
and *e* conſequently the whole CD; *d* therefore likewiſe AE; and ſo meaſures the whole AB. Wherefore d 11. *ax.* 7.
e 12. *ax.* 7.

it is evident that FD is a common measure. If you deny it to be the greatest, let there be a
d 11. ax. 7. greater (G;) then whereas G measures CD, it
e 12. ax. 7. must likewise measure AE, *e* and the residue EB,
g *suppos.* *d* as also CF, *e* and by consequence the residue
h 9. ax. 1. FD, *g* the greater the less. *h Which is absurd.*

Coroll.

Hence, A numcer that measures two numbers, does also measure their greatest common measure.

PROP. III.

A 12
B 8
D 4
C 6
E .. 2
F ---

Three numbers being given, A, B, C, not prime to one another, to find out their greatest common measure E.

Find out D the greatest common measure of the two numbers A, B. If D measures C the third, it is clear that D is the greatest common measure of all the three numbers. If D does not measure C, at least D and C will be composed the one to the other, by the Coroll of the Prop. preceding. Therefore let E be the greatest common measure of the said numbers D and C, and it appears to be the number required.

a *constr.* For E *a* measures C and D, and D measures
b 11. ax. 7. A and B; therefore *b* E measures each of the numbers A, B, C; neither shall any greater (F)
c 11. 1. 7. measure them; for if you affirm that, *c* then F measuring A and B, does likewise measure D their greatest common measure; and in like manner, F measuring D and C, does also mea-
d *suppos.* sure E *c* their greatest common measure, *d* the
e 9. ax. 1. greater the less. *e Which is absurd.*

Coroll.

Hence, A number that measures three numbers, does also measure their greatest common measure.

PROP.

PROP. IV.

A.....6
B.....7
B.....18
B.....9

Every leſs number A is of every greater B either a part or parts.

If A and B be prime to one another, *a* A ſhall be as many parts of the number B, as there are unities in A (as $6 = \frac{6}{7}7$.) But if A meaſures B, it is *b* plain that A is a part of B (as $6 = \frac{1}{3}18$.) Laſtly, if A and B be otherwiſe compoſed to one another, *c* the greateſt common meaſure ſhall determine how many parts A does contain of B; as $6 = \frac{2}{3}9$.

a 4. *def.* 7.
b 3. *def.* 7.
c 4. *def.* 7.

PROP. V.

A.....6 D....4
6 6 4 4
B.....G.....C12. E....H....F8.

If a number A be a part of a number BC, and another number D the ſame part of another number EF; then both the numbers together (A + D) ſhall be the ſame part of both the numbers together (BC + EF,) which one number A is of one number BC.

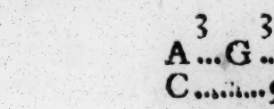
For if BC be reſolved into its parts BG, GC, equal to A; and EF alſo into its parts EH, HF equal to D; *a* the number of parts in BC ſhall be equal to the number of parts in EF. Therefore ſince A + D *b* = BG + EH = GC + HF, thence A + D ſhall be as often in BC + EF, as A is in BC. *Which was to be dem.*

a *hyp.*
b *conſt.* and 2. *ax.* 1.

Or thus. Let $a = \frac{x}{2}$, and $b = \frac{y}{2}$, then 2. a = x, and 2 b = y. wherefore 2 a + 2 b = x + y. therefore $a + b = \frac{x+y}{2}$. *Which was to be dem.*

c 2. *ax.* 1.

PROP. VI.



If a number AB be parts of a number C, and another number DE the same parts of another number F; then both numbers together AB + DE shall be of both numbers together C + F the same parts, that one number AB is of one number C.

Divide AB into its parts AG, GB; and DE into its parts DH, HE. The multitude of parts in both AB, DE is equal by suppoſition; wherefore
a *hyp.* ſince AG *a* is the ſame part of the number C, that DH is of the number F; therefore AG +
b 5. 7. DH *b* ſhall be the ſame part of the compounded number C + F, that one number AG is of one number C. *b* In like manner GB+HE is the ſame part of the ſaid C + F, that one number GB is
c 2. *ax.* 7. of one number C. *c* Therefore AB + DE is the ſame parts of C + F, that AB is of C. *Which was to be dem.*

Or thus. Let $a = \frac{2}{3} x$, and $b = \frac{2}{3} y$, and $x + y = g$, then, becauſe $3a = 2x$, and $3b = 2y$, is $3a + 3b = 2x + 2y = 2g$. therefore $a + b = \frac{2}{3} g = \frac{2}{3} \cdot x + y$.

PROP. VII.

If a number AB be the ſame part of a number CD, that a part taken away AE is of a part taken away CF; then ſhall the reſidue EB be the ſame part of the reſidue FD, that the whole AB is of the whole CD.

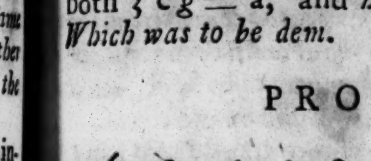
a 1. *poſt.* 7. *a* Let EB be the ſame part of the number GC that
b 5. 7. AB is of CD, or AE of CF. *b* Therefore AE + EB is the ſame part of CF + GC that AE is of CF, or
c 6. *ax.* 1. AB of CD. *c* therefore GF = CD. Take away
d 3. *ax.* 1. CF common to both, and *d* there remains GC =
e 2. *ax.* 7. FD. *e* Wherefore EB is the ſame part of the reſidue FD (GC) that the whole AB is of the whole CB. *Which was to be dem.*

Or

Or thus. Let a + b = x; and c + d = y; and x = 3 y, in like manner as a = 3 c; I ſay b = 3 d. For 3 c + 3 d *f* = 3 y = x *g* = a + b. take away from both 3 c *g* = a, and *h* there remains 3 d = b. *Which was to be dem.*

f 1. 2.
g *hyp.*
h *hyp.*

PROP. VIII.



If a number AB be the ſame parts of a number CD, that a part taken away AE is of a part taken away CF; the reſidue alſo EB ſhall be of the reſidue FD the ſame parts, that the whole AB is of the whole CD.

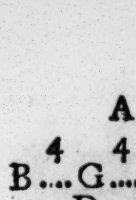
Divide AB into AG, GB, parts of the number CD; alſo AE into AH, HE, parts of the number CF; and take GL = AH = HE. *a* wherefore HG = EL. And becauſe *b* AG = GB, *c* therefore HG = LB. Now whereas the whole AG is the ſame part of the whole CD that the part taken away AH is of the part taken away CF, *d* the reſidue HG or EL ſhall be the ſame part alſo of the reſidue FD that AG is of CD. In like manner, becauſe GB is the ſame part of the whole CD, that HE or GL are of CF, *d* therefore the reſidue LB ſhall be the ſame part of the reſidue FD that GB is of the whole CD. Therefore EL + LB (EB) is the ſame parts of the reſidue FD, that the whole AB is of the whole CD. *Which was to be dem.*

a 3. *ax.* 1.
b *conſtr.*
c 3. *ax.* 1.
d 7. 7.

Or thus more eaſily. Let a + b = x, and c + d = y. Alſo y = $\frac{3}{2}$ x as well as c = $\frac{3}{2}$ a; or, *e* which is the ſame, 3 y = 2 x; and 3 c = 2 a. I ſay d = $\frac{3}{2}$ b. For 3 c + 3 d *f* = 3 y = 2 x *f* = 2 a + 2 b. *g* Therefore 3 c + 3 d = 2 + a 2 b. take away from each 3 c *b* = 2 a. and *b* there remains 3 d = 2 b. *l* therefore d = $\frac{2}{3}$ b. *Which was to be dem.*

e 9. *ax.* 7.
f 1. 2.
g 1. *ax.* 1.
h *hyp.*
k 3. *ax.* 1.
l 8. *ax.*

PROP. IX.



If a number A be a part of a number BC, and another number D the same part of another number EF; then alternately what part or parts the first A is of the third D, the same part or parts shall the second BC be of the fourth EF.

A is supposed ⌐ D. therefore let BG, GC, and EH, HF, parts of the numbers BC, EF be equal; BG and GC to A; and EH, HF to D. The multitude of parts is put equal in both. But it is clear that GB is *a* the same part or parts of EH, that GC is of HF; *b* wherefore BC (BG+GC) is the same part or parts of EF (EH+HF) that BG alone (A) is of EH alone (D.) *Which was to be dem.*

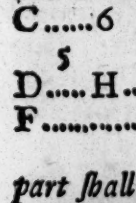
a 1. *ax.* 7. *and* 4. 7.

b 5. & 6. 7.

Or thus. Let $a = \frac{b}{3}$ and $c = \frac{d}{3}$; or $3a = b$ and $\frac{3c}{a} = d$. then is $\frac{c}{a}^{*} = \frac{3c}{3a} = \frac{d}{b}$.

*15. 5.

PROP. X.



If a number AB be parts of a number C, and another number DE the same parts of another number F; then alternately, what part or parts the first AB is of the third DE, the same parts or part shall the second C be of the fourth F.

AB is taken ⌐ DE, and C ⌐ F. Let AG, GB, and DH, HE be parts of the numbers C and F, *viz.* as many in AB, as in DE. It is manifest that AG is the same part of C, that DH is of F. *a* whence alternately AG is of DH, and likewise GB of HE, and *b* so conjointly AB of DE the same part, or parts, that C is of F. *Which was to be dem.*

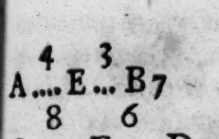
a 9. 7.

b 5. & 9. 7.

Or thus. Let $a = \frac{2b}{3}$, & $c = \frac{2d}{3}$; or $3a = 2b$, & $3c = 2d$. Then is $\frac{c}{a} = \frac{3c}{3a} = \frac{2d}{2b} = \frac{d}{b}$.

PROP.

PROP. XI.



If a part taken away AE be to a part taken away CF, as the whole AB is to the whole CD, the reſidue alſo EB ſhall be to the reſidue FD, as the whole AB is to the whole CD.

Firſt, let AB be ⊐ CD; *a* then AB is either a part or parts of the number CD; and likewiſe AE is *b* the ſame part or parts of CF; *c* therefore the reſidue EB is the ſame part or parts of the reſidue FD, that the whole AB is of the whole CD. *b* and ſo AB. CD :: EB. FD. But if AB be ⊏ CD, then according to what is already ſhewn, will CD. AB :: FD. EB. therefore by inverſion AB. CD :: EB. FD.

a 4. 7.
b 20. *def.* 7.
c 7. or 8. 7.

PROP. XII.

A, 4. C, 2. E, 3.
B, 8. D, 4, F, 6.

If there be divers numbers, how many ſoever, proportional (A. B :: C. D :: E. F;) then as one of the antecedents A is to one of the conſequents B, ſo ſhall all the antecedents (A+C+E) be to all the conſequents (B+D+F.)

Firſt, let A, C, E, be ⊐ B, D, F; then (becauſe of the ſame proportions) *a* ſhall A be the ſame part or parts of B that C is of D; *b* and likewiſe conjointly A+C ſhall be the ſame part or parts of B+D that A alone is of B alone. In like manner A+C+E is the ſame part or parts of B+D+F that A is of B. *c* Therefore A+C+E. B+D+F :: A. B. But if A, C, E, be put greater than B, D, F, the ſame may be ſhewn by inverſion.

a 20. *def.* 7.
b 5. & 6. 7
c 20 *def.* 7.

PROP. XIII.

A, 3. C, 4. B, 5. D, 12. *If there be four numbers proportional (A.B :: C. D.) then alternately they shall also be proportional, (A.C :: B.D.)*

First let A and C be ⌐ B and D, and A ⌐ C. By reason of the same proportion *a* shall A be the same part or parts of B, that C is of D. *b* Therefore alternately A is the same part or parts of C that B is of D. and so A. C :: B. D. But if A be ⌐ C, and A and C supposed ⌐ B and D, the matter will be the same by inverting the proportions.

a 20. *def.* 7.
b 9. & 10. 7.

PROP. XIV.

A, 9. D, 6. B, 6. E, 4. C, 3. F, 2. *If there be numbers, how many soever, A, B, C, and as many more equal to them in multitude, which may be compared two and two in the same proportion (A.B :: D. E. and B.C :: E.F;) they shall also, of equality, be in the same proportion (A. C :: D.F.)*

For because A. B :: D. E. *a* therefore alternately is A. D :: B. E :: *a* C. F. *a* and so again by changing A. C :: D. F. *Which was to be dem.*

a 13. 7.

PROP. XV.

1. D.. B... 3. E..... 6. *If a unit measure any number B, and another number D do equally measure some other number E; alternately also shall a unit measure the third number D, as often as the second B does the fourth E.*

For seeing 1 is the same part of B, that D is of E; *a* therefore alternately shall 1 be the same part of D, that B is of E. *Which was to be demonstrated.*

a 9. 7.

PROP.

PROP. XVI.

B, 4. A, 3. *If two numbers, A, B, multi-*
A. 3. B, 4. *plying themſelves the one into*
AB, 12. BA, 12. *the other, produce any numbers AB, BA; the numbers produced AB and BA ſhall be equal the one to the other.*

For becauſe AB = A × B, *a* therefore ſhall 1 be as often in A, as B in AB, *b* and by conſequence alternately 1 ſhall be as often in B as A in AB. But for that BA = B × A, *a* therefore ſhall 1 be as often in B, as A in BA. therefore as often as 1 is in AB, ſo often is 1 in BA. and *c* ſo AB = BA. *Which was to be dem.*

a 15. *def.* 7.
b 15. 7.
c 4. *ax.* 7.

PROP. XVII.

A, 3. *If a number A multiplying*
B, 2. C, 4. *two numbers B, C, produce other*
AB, 6. AC, 12. *numbers, AB, AC; the numbers produc'd of them ſhall be in the ſame proportion that the numbers multiplied are.* (*AB. AC :: B. C.*)

For being AB = A × B, *a* therefore ſhall 1 be as often in A, as B in AB. *a* Likewiſe becauſe AC = A × C, therefore ſhall 1 be as often in A, as C in AC. and ſo alſo B as often in AB as C in AC. *b* wherefore B. AB :: C. AC. *c* and therefore alſo alternately B. C :: AB. AC. *Which was to be dem.*

a 15. *def.* 7.
b 20. *def.* 7.
c 13. 7.

PROP. XVIII.

C, 5. C, 5. *If two numbers A, B, mul-*
A, 3. B, 9. *tiplying any number C, produce*
AC, 15. BC, 45. *other numbers AC, BC; the numbers produced of them ſhall be in the ſame proportion that the numbers multiplying are.* (*A. B :: AC. BC.*)

For *a* AC = CA, and BC *a* = CB; ſo the ſame C multiplying A and B produces AC and BC. *b* therefore A. B :: AC. BC. *Which was to be dem.*

a 16. 7.
b 17. 7.

Schol.

Schol.

Hence is deduced the vulgar manner of reducing fractions ($\frac{3}{5}$, $\frac{7}{9}$) to the same denomination. For multiply 9 both by 3 and 5, and they produce $\frac{27}{45} = \frac{3}{5}$; because by this, 3. 5 :: 27. 45. Likewise multiply 5 by 7 and 9; there arises $\frac{35}{45} = \frac{7}{9}$; because 7. 9 :: 35. 45.

PROP. XIX.

A, 4. B, 6. C, 8. D, 12. AD, 48. BC, 48.

If there be four numbers in proportion (A. B :: C. D) the number produced of the first and fourth (AD) is equal to the number which is produced of the second and third (BC.) And if the number which is produced of the first and fourth (AD) be equal to that produced of the second and third (BC) those four numbers shall be in proportion (A. B :: C. D.)

a 17. 7. b *hyp.* c 18. 7. d 9. 5. e *hyp.* f 7. 5. g 17. 7. h 18. 7. k 11. 5.

1. *Hyp.* For AC. AD *a* :: C. D *b* :: A. B *c* :: AC. BC. *d* therefore AD = BC. *Which was to be dem.*

2. *Hyp.* Because *e* AD = BC, therefore AC. AD *f* :: AC. BC. But AC. AD *g* :: C. D. and AC. BC *h* :: A. B. *k* therefore C. D :: A. B. *Which was to be demonstrated.*

PROP. XX.

A. 4. B. 6. C. 9. AC, 36. BB, 36. D, 6.

If there be three numbers in proportion (A. B :: B. C) the number contained under the extremes (AC) is equal to the square made of the middle (BB.) And if the number contained under the extremes be equal to that (Bq.) produced of the middle, those three numbers shall be in proportion $\left(\frac{A}{B} : \frac{B}{C}\right)$

a 1. *ax.* 7. b 19. 7.

1. *Hyp.* For take D = B. *a* therefore A. B :: D (B.) C. *b* wherefore AC = BD, *a* or BB. *Which was to be demonstrated.*

2. *Hyp.*

2. *Hyp.* Becauſe AC *c* = BD, *d* therefore A. B :: D (B) C. *Which was to be dem.*

c *hyp.*
d 19. 7.

PROP. XXI.

A ... G .. B 5. E 10
C .. H. D 3. F 6

Numbers AB, CD, being the leaſt of all that have the ſame proportion with them (E, F) do equally meaſure the numbers, E, F, having the ſame proportion with them; the greater AB the greater E, and the leſſer CD the leſſer F.

For AB. CD *a* :: E. F. *b* therefore alternately AB. E :: CD. F. *c* therefore AB is the ſame part or parts of E that CD is of F. Not parts; for if ſo, then let AG, GB be parts of the number E; and CH, HD, parts of the number F. *e* therefore AG. E :: CH. F. and by inverſion AG. CH *d* :: E. F *e* :: AB. CD. therefore AB, CD are not the leaſt in their proportion; which is contrary to the Hypotheſis. Therefore, *&c.*

a *hyp.*
b 13. 7.
c 20. *def.* 7.
d 13. 7.
e *hyp.*

PROP. XXII.

A, 4. D, 12.
B, 3. E, 8.
C, 2. F, 6.

If there be three numbers A, B, C; and other numbers equal to them in multitude, D, E, F; which may be compared two and two in the ſame proportion: And if alſo the proportion of them be perturbed (A. B :: E. F. and B. C :: D. E.) then of equality they ſhall be in the ſame proportion (A. C :: D. F.)

For becauſe A, B *a* :: E. F, therefore ſhall AF = BE; and becauſe B. C :: *a* D. E, *b* therefore BE = CD. *c* and conſequently AF = CD. *d* Wherefore A. C :: D. F. *Which was to be dem.*

a *hyp.*
b 19. 7.
c 1. *ax.* 1.
d 19. 7.

PROP.

PROP. XXIII.

A, 9. B, 4.
C---- D---
E--

Numbers prime the one to the other, A, B, are the leaſt of all numbers that have the ſame proportion with them.

If it be poſſible, let C and D be leſs than A and B, and in the ſame proportion; *a* therefore C meaſures A equally as D meaſures B, namely, by the ſame number F; and ſo C ſhall be *b* as often in A as 1 is in E; *c* likewiſe alternately, E as often in A as 1 in C. By the like inference as many times as 1 is in D, ſo many times ſhall E be in E. Therefore E meaſures both A and B; which conſequently are not prime the one to the other, *contrary to the Hypotheſis.*

a 21. 7.
b 23. *def.* 7.
c 15. 7.

PROP. XXIV.

A, 9. B, 4.
C---
D--- E--

Numbers A, B, being the leaſt of all that have the ſame proportion with them, are prime the one to the others.

If it be poſſible, let A and B have a common meaſure C; and let the ſame meaſure A by D, and B by E; *a* therefore CD=A, *b* and CE=B. *b* Wherefore A. B:: D. E. But D and E are leſſer than A and B, as being but parts of them. Therefore A and B are not the leaſt in their proportion, *againſt the Hypotheſis.*

a 9. *ax.* 7.
b 17. 7.

PROP. XXV.

A, 9. B, 4.
C, 3. D--

If two numbers A, B, be prime the one to the other, the number C meaſuring one of them A, ſhall be prime to the other number B.

For if you affirm any other D to meaſure the numbers B and C, *a* then D meaſuring C does alſo meaſure A; and conſequently A and B are not prime the one to the other. *Which is againſt the Hypotheſis.*

a 11. *ax.* 7.

PROP.

PROP. XXVI.

A, 5. C, 8. B, 3. AB, 15. E - - - - F - - - -

If two numbers, A, B, be prime to any number C, the number alſo produced of them A, B ſhall be prime to the ſame C.

If it be poſſible, let the number E be a common meaſure to AB, and C; and let $\frac{AB}{E}$ be = F; *a* thence AB = EF; *b* wherefore alſo E. A :: B. F. But becauſe A is prime to C, which E meaſures, *c* therefore E and A are prime to one another, *d* and ſo leaſt in their own proportion, *e* and conſequently they muſt meaſure B and F; namely F ſhall meaſure B, and A ſhall meaſure F. Therefore ſeeing E meaſures both B and C, they ſhall not be prime to one another: *Contrary to the Hypotheſis.*

a 9. *ax.* 7.
b 19. 7.
c 25. 7.
d 23. 7.
e 21. 7.

PROP. XXVII.

A, 4. B, 5. Aq, 16. D, 4.

If two numbers A, B, be prime to one another, that alſo which is produc'd of one of them (Aq) ſhall be prime to the other B.

Take D = A; therefore each of D, and A are prime to B. *b* wherefore AD or Aq is prime to B. *Which was to be dem.*

a 1. *ax.* 7.
b 26. 7.

PROP. XXVIII.

A, 5. C, 4. B, 3. D, 2. AB, 15. CD, 8.

If two numbers A, B, be prime to two numbers C, D, each to either of both, the numbers alſo produced of them AB, CD ſhall be prime to one another.

For being A and B are prime to C, *a* therefore ſhall AB alſo be prime to the ſame. And by the

a 26. 7.

ſame

b 26. 7. ſame reaſon ſhall AB be prime to D. *b* Therefore AB is prime to CD. *Which was to be dem.*

PROP. XXIX.

A, 3. B, 2. Aq, 9. Bq, 4. Ac, 27. Bc, 8.

If two numbers A, B, be prime to one another, and each multiplying himſelf produce another number (Aq, and Bq;) *then the number produced of them* (Aq, Bq) *ſhall be prime to one another. And if the numbers given at firſt, A, B, multiplying the ſaid produced numbers* (Aq, Bq) *produce others* (Ac, Bc) *thoſe numbers alſo ſhall be prime to one another: And this ſhall ever happen about the extremes.*

a 27. 7. For becauſe A is prime to B, *a* therefore Aq ſhall be prime to B. and Aq being prime to B, *a* therefore Aq ſhall alſo be prime to Bq. Again, becauſe A is as well prime to B and Bq. as Aq
b 28. 7. is to the ſaid B and Bq, *b* therefore ſhall A × Aq, that is, Ac, be prime to B × Bq, that is, to Bc: And ſo forth of the reſt.

PROP. XXX.

8 5
A..... B..... C 13 D----

If two numbers AB, BC, be prime the one to the other, then both added together (AC) *ſhall be prime to either of them AB, BC. And if both added together AC be prime to any one of them AB, the numbers alſo given in the beginning AB, BC ſhall be prime to one another.*

1. *Hyp.* For if you would have AC, AB to be
a 12. *ax.* 7. compoſed, let D be the common meaſure: *a* this ſhall meaſure the reſidue BC: and therefore AB, BC, are not prime to one another; *which is againſt the Hypotheſis.*

2. *Hyp.* AC, AB being taken for prime to one another, let D be the common meaſure of AB,
b 10. *ax.* 7. BC. *b* But ſeeing that meaſures the whole AC, therefore AC, AB, are not prime to one another; *contrary to the Hypotheſis.*

Coroll.

Coroll.

Hence, A number which being compounded of two, is prime to one of them, is alſo prime to the other.

PROP. XXXI.

A, 5. B, 8. *Every prime number A is prime to every number B, which it meaſures not.*

For if any common meaſure does meaſure both A, B, *a* then A will not be a prime number; *contrary to the Hypotheſis.*

a 11. *def.* 7.

PROP. XXXII.

A, 4. D, 3. B, 6. E, 8. AB, 24. *If two numbers A, B, multiplying one another produce another AB, and ſome prime number D meaſure the number produced of them AB; then ſhall it alſo meaſure one of thoſe numbers, A or B, which were given at the beginning.*

Suppoſe the number D not to meaſure the number A, and let $\frac{AB}{D}$=E. *a* then AB=DE, *b* whence D. A :: B. E. *c* But D is prime to A; *d* therefore D and A are the leaſt in their proportion; *e* and conſequently D meaſures B as often as A meaſures E. *Which was to be dem.*

a 9. *ax.* 7.
b 19. 7.
c *hyp. and* 31. 7.
d 23. 7.
e 21. 7.

PROP. XXXIII.

A, 12. B, 2. *Every compoſed number A is meaſured by ſome prime number B.*

Let one or more numbers *a* meaſure A, of which let the leaſt be B; that ſhall be a prime number: For if it be ſaid to be compoſed, then ſome *a* leſſer number ſhall meaſure it, *b* which ſhall alſo conſequently meaſure A. Wherefore B is not the leaſt of them which meaſure A, *contrary to the Hyp.*

a 13. *def.* 7.
b 11. *ax.* 7.

PROP.

PROP. XXXIV.

A, 9. *Every number A, is either a prime, or measured by some prime number.*

For A is either necessarily a prime or a composed number. If it be a prime, 'tis that we affirm. If composed, *a* then some prime number measures it. *Which was to be dem.*

a 33. 7.

PROP. XXXV.

A, 6. B, 4. C, 8. D, 2. E, 3. F, 2. G, 4. H--I--K---- L---

How many numbers soever A, B, C, being given, to find the least numbers E, F, G, that have the same proportion with them.

a 23. 7. If A, B, C, be prime to one another, *a* they
b 3. 7. shall be the least in their proportion. If they be composed, *b* let their greatest common measure be D, which let measure them by E, F, G. These are then least in the proportion A, B. C.
c 9. *ax.* 7. For D × E, F, G, *c* produces ABC, *d* therefore
d 17. 7. they are all in the same proportion. But allow other numbers H, I, K to be the least in the
e 21. 7. same proportion; *e* which shall therefore equally
f 9. *ax.* 7. measure ABC, namely by the number L. *f* there-
g 1. *ax.* 1. fore L × H,I,K, shall produce A,B,C, *g* and con-
h 19. 7. sequently ED=A=HL. *h* from whence E. H:: L. D. But E *k* ⊏ K; *l* therefore L ⊏ D, and so
k *suppos.* D is not the greatest common measure of A,
l 20. *def.* 7. B, C. *Which is against the Hypothesis.*

Coroll.

Hence, The greatest common measure of how many numbers soever, does measure them by the numbers which are least of all that have the same proportion with them. Whereby appears the vulgar method of reducing fractions to the least terms.

PROP.

PROP. XXXVI.

Two numbers being given A, B, to find out the leaſt number which they meaſure.

A, 5. B, 4. AB, 20. D------ E---F--- 1. *Caſe.* If A and B be prime the one to the other, AB is the number required. For it is manifeſt that A and B meaſure AB. If it be poſſible, let A and B meaſure ſome other number D ⌐ AB, if you pleaſe by E, and F. *a* therefore AE = D = BF, *b* and ſo A. B :: F. E. But becauſe A and B *c* are prime the one to the other, *d* and ſo leaſt in their proportion, A ſhall *e* equally meaſure F as B does E. But B. E *f* :: AB. AE (D) *g* Therefore AB ſhall alſo meaſure D, which is leſs than it ſelf. *Which is abſurd.*

a 9. *ax.* 7. & 1. *ax.* 1.
b 19. 7.
c *hyp.*
d 23. 7.
e 21. 7.
f 17. 7.
g 20. *def.* 7.

A, 6. B, 4. C, 3. D, 2. AD, 12. F---- G--H--- 2. *Caſe.* But if A and B be compoſed one to another, *h* let there be found C and D the leaſt in the ſame proportion. *k* therefore AD = BC; and AD or BC ſhall be the number ſought for.

h 35. 7.
k 19. 7.

For it is *l* plain that B and D do meaſure AD or BC. Conceive A and B to meaſure F ⌐ AD, namely A by G, and B by H. *m* therefore AG = F = BH. *n* whence A. B :: H. G *o* :: C. D. *p* and conſequently C equally meaſures H as D does G. But D. G *q* :: AD. AG (F.) therefore AD *r* meaſures F, the greater the leſs. *Which is abſurd.*

l 7. *ax.* 7.
m 9. *ax.* 7.
n 19. 7.
o *conſtr.*
p 21. 7.
q 17. 7.
r 20. *def.* 7.

Coroll.

Hence, If two numbers multiply the leaſt that are in the ſame proportion, the greater the leſs, and the leſs the greater, the leaſt number which they meaſure ſhall be produced.

PROP. XXXVII.

A, 2. B, 3. E.....6 C----F----D

If two numbers A, B, meaſure any number C, D, the leaſt number which they meaſure E ſhall alſo meaſure the ſame CD.

If you deny it, take E from CD as often as you can, and leave FD ⊐ E. therefore ſeeing A and B *a* meaſure E, *b* and E meaſures CF, likewiſe A and B will meaſure CF. But *a* they meaſure the whole CD; *d* therefore alſo they meaſure the reſidue FD; and conſequently E is not the leaſt which A and B meaſure; *contrary to the Hypotheſis.*

a *hyp.*
b *conſtr.*
c 11.*ax.*7.
d 12.*ax.*7.

PROP. XXXVIII.

A, 3. B, 4. C,6. D, 12.

Three numbers being given A, B, C, to find out the leaſt which they meaſure.

a 36. 7. *a* Find D to be the leaſt that two of them A and B do meaſure; which if the third C do alſo meaſure, it is manifeſt that D is the number ſought for. But if C do not meaſure D, let E be the leaſt that C and D do meaſure. E ſhall be the number required.

A, 2. B, 3. C, 4. D, 6. E, 12. F---

For it appears by the 11. *ax.* 7. that A, B, C meaſure E; and it is eaſily ſhewn that they meaſure no other leſs than F. For if you affirm they do, *b* then D meaſures F, *b* and conſequently E meaſures the ſame F, the greater the leſs. *Which is abſurd.*

b 37. 7.

Coroll.

Hence it appears, that if three numbers meaſure any number, the leaſt alſo, which they meaſure, ſhall meaſure the ſame.

PROP.

PROP. XXXIX.

A, 12. B, 4. C, 3. *If any number B measure a number A, the number measured A, shall have a part C denominated of the number measuring B.*

For because $\frac{A}{B} a = C$, *b* shall $A = BC$. *c* therefore $\frac{A}{C} = B$. *Which was to be dem.*

a *hyp.* b 9. *ax.* 7. c 7. *ax.* 7.

PROP. XL.

A, 15. B, 3. C, 5. *If a number A have any part whatsoever B, the number C, from which the part B is denominated, shall measure the same.*

For being $BC\ a = A$, *b* thence $\frac{A}{C} = B$. *Which was to be dem.*

a *hyp.* & 9. *ax.* 7. b 7. *ax.* 7.

PROP. XLI.

$\frac{1}{2}$ G, 12. $\frac{1}{3}$ H--- $\frac{1}{4}$ *To find out a number G, which being the least, contains the parts given* $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.

a Let G be found the least which the denominators, 2, 3, 4, measure; *b* it is evident that G has the parts $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. If it be possible let H ⊐ G have the same parts; *c* therefore 2, 3, 4, measure H; and so G is not the least which 2, 3, 4 measure: *against the constr.*

a 38. 7. b 39. 7. c 40. 7.

The End of the seventh Book.

THE EIGHTH BOOK OF EUCLIDE's ELEMENTS

PROP. I.

A, 8. B, 12. C, 18. D, 27.
E-F--G---H----

IF there be divers numbers how many ſoever in continual proportion, A,B,C,D, and their extremes A,D, prime to one another; then thoſe numbers A,B,C,D, are the leaſt of all numbers that have the ſame proportion with them.

For, if it be poſſible, let there be as many others E, F, G, H, leſs than A, B, C, D, and in the ſame proportion with them. *a* Therefore *ex* equality A. D :: E. H. and conſequently A and D are prime numbers, *b* and ſo the leaſt in their proportion, *c* equally meaſuring E and H, which are leſs than themſelves. *Which is abſurd.*

a 14. 7.
b 23. 7.
c 21. 7.

PROP. II.

1.
A, 2. B, 3.
Aq, 4. AB, 6. Bq, 9.
Ac, 8. AqB, 12. ABq, 18. Bc, 27.

To find out the leaſt numbers continually proportional, as many as ſhall be required, in the proportion given of A to B.

Let A and B be the leaſt in the proportion given; then Aq, AB, Bq, ſhall be three laſt in the ſame continual proportion that A is to B.

For

For AA. AB *a* :: A. B *a* :: AB. BB. Likewiſe a 17. 7.
becauſe A and B are *b* prime to one another, *c* b 24. 7.
ſhall Aq, Bq, be alſo prime to one another, *d* c 29. 7.
and ſo Aq, AB, Bq, are ∴ the leaſt in the pro- d 1. 8.
portion of A to B.

Moreover, I ſay Ac, AqB, ABq, Bc, are the four leaſt in the proportion of A to B. For AqA. AqB *e* :: A. B. *e* :: ABA (AqB.) ABB. *e* and A. e 17. 7.
B :: ABq. BBq (Bc.) Therefore ſince Ac, and Bc, f 29. 7.
are *f* prime to one another, likewiſe *g* ſhall Ac, g 1. 8.
AqB, ABq, Bc be the four leaſt ∴ in the proportion of A to B. In the ſame manner may you find out as many proportional numbers as you pleaſe. *Which was to be done.*

Coroll.

1. Hence, If three numbers, being the leaſt, are proportional, their extremes ſhall be ſquares; if four, cubes.

2. How many extremes proportional ſoever there be, being by this prop. found to be the leaſt in the given proportion, they are prime to one another.

3. Two numbers, being the leaſt in the given proportion, do meaſure all the mean numbers whatſoever of the leaſt in the ſame proportion; becauſe they ariſe from the multiplication of them into certain other numbers.

4. Hence alſo it appears by the conſtruction, that the ſeries of numbers 1, A, Aq, Ac; 1, B, Bq. Bc; Ac, AqB, ABq, Bc conſiſts of an equal multitude of numbers; and conſequently, the extreme numbers of how many ſoever the leaſt continually proportionals are the laſt of as many other continually proportionals from a unite, as the extreme Ac, Bc, of the continually proportionals Ac, AqB. ABq, Bc, are the leaſt of as many proportionals from a unite, 1, A, Aq, Ac; and 1, B, Bq, Bc.

5. 1, A, Aq, Ac; and B, BA, BAq; and Bq, ABq, are ∴ in the proportion of 1 to A. Alſo B, Bq, Bc; and A, AB, ABq; and Aq, AqB are ∴ in the proportion of 1 to B.

PROP. III.

A, 8. B, 12. C, 18. D, 28. *If there be numbers continually proportional, how many ſoever, A, B, C, D, being alſo the leaſt of all that have the ſame proportion with them; their extremes A, D, are prime to one another.*

a 2. 8. For if there be *a* found as many numbers the leaſt in the proportion of A to B, they ſhall be no other than A, B, C, D; therefore, by the ſecond *Coroll.* of the precedent *Prop.* the extremes A and D are prime to one another. *Which was to be dem.*

PROP. IV.

A, 6. B, 5. C, 4. D, 3.
H, 4. F, 24. E, 20. G, 15.
I -- K --- L ---

Proportions how many ſoever being given, in the leaſt numbers (A. to B, and C to D,) to find out the leaſt numbers continually proportional in the proportions given.

a 36. 7. *a* Find out E the leaſt number which B and C
b 3. *poſt.* 7. do meaſure; and let B meaſure E *b* as often as A does another F, *viz.* by the ſame number H. *b* Alſo let C meaſure the ſaid E as often as D meaſures another G. then F, E, G, ſhall be the leaſt
c 9. *ax.* 7. in the proportions given. For AH *c* = F, and
d 18. 7. BC *c* = E; *d* therefore A. B :: AH. BH *c* :: F. E.
e 7. 5. In like manner C. D :: E. G; therefore F, E, G, are continually proportional in the proportions given. And they are moreover the leaſt in the ſaid proportions: For conceive other numbers,
f 21. 7. I, K, L, to be the leaſt; *f* then A and B muſt equally meaſure I and K, *f* and C and D likewiſe K and L; and ſo B and C meaſure the ſame K.
g 37. 7. *g* Wherefore alſo E meaſures the ſame number K, which is leſs than it ſelf. *Which is abſurd.*

A, 6.

A,6. B,5. C,4. D,3. E,5. F,7.
H,24. G,20. I,15. K,21.

But three proportions being given, A to B, C to D, and E to F; find out as before three numbers H, G, I, the leaſt continually in the proportions of A to B, and C to D. Then if E meaſures *b* take another number K which may be equal- h 3.*poſt*.7.
ly meaſur'd by F; and thoſe four numbers H,G, I, K, ſhall be continually the leaſt in the given proportions; which we need go no other way to prove than we did in the firſt part.

A,6. B,5. C,4. D,3. E,2. F,7.
H,24. G,29. I,25.
M,48. L,40. K,30. N,105.

If E do not meaſure I, let K be the leaſt which E and I do meaſure; and as often as I meaſures K, let G as often meaſure L, and H alſo M. ſo likewiſe let F meaſure N as often as E meaſures K. The four numbers, M, L, K, N ſhall be leaſt, continually in the given proportions; which we may demonſtrate as before.

PROP. V.

C,4. E,3. D,6. F,16. CD,24. EF,48. ED,18.

Plane numbers CD, EF, are in that proportion to one another which is compoſed of their ſides.

For becauſe CD. ED *a* :: C.E; *a* and ED. EF :: a 17.7.
D. F. & $\frac{CD}{EF}$ *b* = $\frac{CD}{ED} + \frac{D}{E}$ *c* then ſhall be the pro- b 20.*def*.6
portion $\frac{CD}{EF} = \frac{C}{E} + \frac{D}{F}$. *Which was to be dem.* c 11. 5.

PROP. VI.

A,16. B,24. C,36. D,54. E,81. F,4. G,6. H,9.

If there be numbers continually proportional how many ſoever, A, B, C, D, E, and the firſt A do not meaſure the ſecond B, neither ſhall any of the other meaſure any one of the reſt.

a 20.*def*.7. Becauſe A does not meaſure B, *a* neither ſhall any one meaſure that which next follows; be-
b 35. 7. ing A.B :: B.C :: C.D. *&c.* *b* Take three numbers, F, G, H, the leaſt in the proportion of A to B, therefore ſince A does not meaſure B, *a* neither
c 5. *ax*. 7. ſhall F meaſure G. *c* therefore F is not a unit. But
d 3. 3. F and H are prime to one another; and ſo, *d* be-
e 14. 7. ing of equality A. C :: F.H. and F does not meaſure H, *a* neither ſhall A meaſure C; and conſequently neither ſhall B meaſure D, nor C meaſure E, *&c.* becauſe A. C *e* :: B. D *e* :: C.E, *&c.* In like manner four or five numbers being taken the leaſt in the proportion of A to B, it will appear that A does not meaſure D and E; nor does B meaſure E and F, *&c.* Wherefore none of them ſhall meaſure any other. *Which was to be dem.*

PROP. VII.

A, 3. B, 6. C, 12. D, 24. E, 48.

If there be numbers continually proportional how many ſoever A, B, C, D, E, and the firſt A meaſure the laſt E, it ſhall alſo meaſure the ſecond B.

a 6. 7. If you deny that A meaſures B, *a* then neither ſhall it meaſure E; *Which is contrary to the Hyp.*

PROP. VIII.

A, 24. C, 36. D, 54. B, 81.
G, 8. H, 12. I, 18. K, 27.
E, 32. L, 48. M, 72. F, 108.

If between two numbers A, B, there fall mean proportional numbers in continual proportion C, D; as many mean continually proportional numbers as fall between them, ſo many alſo mean continually proportional numbers ſhall fall between two other numbers E, F, which have the ſame proportion with them. (L, M.)

a 35. 7. *a* Take G, H, I, K, the leaſt ∴ in the proportion
b 14. 7. of A to C; *b* of equality ſhall G.K :: A.B *c* :: E.F.
c *hyp.* But G, and K *d* are prime to one another. *e*
d 3. 8. Wherefore G meaſures E as often as K does F.
e 21. 7. Let H meaſure L, and I likewiſe M by the ſame
f *conſtr.* number. *f* therefore E, L, M, F, are in ſuch proportion as G, H, I, K, that is as A, B, C, D. *Which was to be dem.*

PROP.

PROP. IX.

1. E,2. F,3. G,4. H,6. I,9. A,8. C,12. D,18. B,17.

If two numbers A, B be prime to one another, and mean numbers in continual proportion C,D fall between them; as many mean numbers in continual proportion as fall between them, so many means also (E, G; and F. I) *shall fall in continual proportion between either of them and a unity.*

It is evident that 1, E, G, A, and 1, F, I, B, are ∷, and as many as A, C, D, B, namely by the 4. Coroll. 2. 8. *Which was to be dem.*

PROP. X.

A.8. I,12. K,18. B,27. E,4. DF,6. G,9. D,2. F,3. 1.

If between two numbers A,B, and a unit, numbers continually proportional (E, D, and F, G,) *do fall, how many mean numbers in continual proportion fall between either of them and a unit, so many means also shall fall in continual proportion between them, I, K.*

For E, DF, G, and A, DqF (I) DG, (K,) B, are ∷ by 2. 8. therefore, *&c.*

PROP. XI.

A,2. B,3. Aq,4. AB,6. Bq,9.

Between two square numbers, Aq, Bq, there is one mean proportional number AB: and the square Aq to the square Bq is in double proportion of that of the side A to the side B.

a It is manifest that Aq, AB, Bq, are ∷; *b* and consequently also $\frac{Aq}{Bq} = \frac{A}{B}$ doubly. *W. W. to be dem.*

a 17. 7.
b 10. *def.* 7.

PROP.

PROP. XII.

Ac,27. AqB,36. ABq,48. Bc,64. A, 3. B, 4. Aq, 9. AB, 12. Bq, 16.

Between two cube numbers, Ac, Bc, there are two mean proportional numbers AqB, ABq: and the cube Ac is to the cube Bc in treble proportion of that in which the side A is to the side B.

a 2. 8. b 10. *def.* 5. *a* For Ac, AqB, ABq, Bc, are ∷ in the proportion of A to B; *b* and therefore $\frac{Ac}{Bc} = \frac{A}{B}$ trebly.

Which was to be dem.

PROP. XIII.

A, 2. B, 4. C, 8.
Aq, 4. AB, 8. Bq, 16. BC, 32. Cq, 64.
Ac, 8. AqB, 16. ABq, 32. Bc, 64. BqC, 128. BCq, 256. Cc, 512.

If there be numbers in continual proportion how many soever A, B, C; and every of them multiplying it self produce certain numbers; the numbers produced of them Aq, Bq, Cq, shall be proportional: And if the number first given A, B, C, multiplying their products Aq, Bq, Cq, produce other numbers, Ac, Bc, Cc, they also shall be proportional; and this shall ever happen to the extremes.

a 2. 8. b 14. 7. For Aq, AB, Bq, BC, Cq *a* are ∷ *b* therefore of equality Aq. Bq :: Bq. Cq. *Which was to be dem.*

a Also Ac, AqB, ABq, Bc, BqC, BCq, Cc, are ∷; *b* therefore likewise of equality Ac. Bc :: Bc. Cc. *Which was to be dem.*

PROP. XIV.

Aq, 4. AB, 12. Bq, 36. A, 2. B, 6.

If a square number Aq measure a square number Bq, the side also of the one (A) shall measure the side of the other (B:) and if the side of one square A measure the side of another B, the square Aq shall likewise measure the square Bq.

1. *Hyp.*

1. *Hyp.* For Aq. AB *a* :: AB. Bq. therefore seeing by the Hypotheſis Aq meaſures Bq, *b* it ſhall meaſure alſo AB. But Aq. AB :: A. B. *c* therefore A meaſures B. *Which was to be dem.* a 2. & 11. 8 b 7. 8. c 20. *def.* 7.

2. *Hyp.* A meaſures B. *c* therefore Aq ſhall as well meaſure AB, *c* as AB meaſures Bq; *d* and conſequently Aq meaſures Bq. *Which was to be dem.* d 11. *ax.* 7.

PROP. XV.

A, 2. B, 6. Ac, 8. AqB, 24. ABq, 72. Bc, 216.

If a cube number Ac meaſures a cube number Bc, then the ſide of the one (A) ſhall meaſure the ſide of the other (B:) And if the ſide A of one cube Ac meaſure the ſide B of the other Bc, alſo the cube Ac ſhall meaſure the cube Bc.

1. *Hyp.* For Ac, AqB, ABq, Bc *a* are ∴, therefore Ac, *b* meaſuring the extreme Bc. ſhall alſo *c* meaſure the ſecond AqB. But Ac. Aq. B :: A. B. *d* therefore A ſhall alſo meaſure B. a 2. & 12. 8 b *hyp.* c 7. 8. d 20. *def.* 7.

2. *Hyp.* A meaſures B; *d* therefore Ac meaſures AqB, which alſo meaſures ABq, and that Bc; *e* therefore Ac ſhall meaſure Bc. *Which was to be dem.* e 11. *ax.* 7.

PROP. XVI.

Aq, 4. B, 9. Aq, 16. Bq, 81.

If a ſquare number Aq do not meaſure a ſquare number Bq, neither ſhall the ſide of the one A meaſure the ſide of the other B: And if A the ſide of the one ſquare Aq do not meaſure B the ſide of the other Bq, neither ſhall the ſquare Aq meaſure the ſquare Bq.

1. *Hyp.* For if you affirm that A meaſures B, *a* then Aq alſo ſhall meaſure Bq. *againſt the Hyp.* a 14. 8.

2. *Hyp.* If you maintain Aq to meaſure Bq; *a* then likewiſe A ſhall meaſure B. *contrary to the Hypotheſis.*

PROP.

PROP. XVII.

A,2. B,3. Ac,8. Bc,27. *If a cube number Ac do not measure a cube number Bc, neither shall the side of one A, measure the side of the other B: And if A the side of one cube Ac do not measure B the side of the other Bc, neither shall the cube Ac measure the cube Bc.*

a 15. 8. 1. *Hyp.* Let A measure B; *a* then Ac shall measure Bc. *against the Hyp.*

2. *Hyp.* Let Ac measure Bc; then A shall measure B; *which is also against the Hypothesis.*

PROP. XVIII.

C,6. D,2. CD,12. E,9. F,3. DE,18. EF,27. *Between two like plane numbers CD and EF there is one mean proportional number DE: And the plane CD is to the plane EF in double proportion of that which the side C has to the homologous side (or of like proportion) E.*

* 21. *def.* 7. a 17. 7. b 11. 5. c 10. *def.* 5. Being * by the Hypoth. C. D :: E. F. therefore by inversion C. E :: D. F. But C.E *a* :: CD. DE; *a* and D.F :: DE.EF. *b* therefore CD.DE :: DE. EF. *c* Wherefore the proportion of CD to EF is double to that of CD to DE, that is, to the proportion of C to E, or D to F.

Coroll.

Hence it is apparent, that between two like plane numbers there falls one mean proportional in the proportion of the homologous sides.

PROP.

PROP. XIX.

CDE, 30. DEF, 60. FGE, 120. FGH, 240.
CD, 6. DF, 12. FG, 24.
C, 2. D, 3. E, 5. F, 3. G, 6. H, 10.

Between two like ſolid numbers CDE, FGH, there are two mean proportional numbers DFE, FGE. And the ſolid CDE is to the ſolid FGH, in treble proportion of that which the homologous ſide C has to the homologous ſide F.

Whereas by the * hyp. C.D :: F.G, & D.E :: G. H. therefore *a* by inverſion ſhall C. F :: D. G :: *a* E.H. But CD. DF *b* :: C. F, and DF. FG *b* :: D. G; *c* wherefore CD. DF :: DF. FG :: E.H. *d* and accordingly CDE. DFE :: DFE. FGE :: E. H :: FGE. FGH. Therefore between CDE. FGH, fall two mean proportionals DFE, FGE. *e* And ſo it is plain that the proportion of CDE to FGH is treble to that of CDE to FDE, or C to F. *Which was to be dem.*

* 21. *def.* 7.
a 13. 7.
b 17. 7.
c 11. 5.
d 17. 7.
e 10. *def.* 5.

Coroll.

Hereby it is manifeſt, that between two like ſolid numbers there fall two mean proportionals in the proportion of the homologous ſides.

PROP. XX.

A, 12. C, 18. B, 27. D, 2. E, 3. F, 6. G, 9.

If between two numbers A, B, there fall one mean proportional number C; theſe numbers A, B, are like plane numbers.

a Take D and E the leaſt in the proportion of A to C, or C to B. then D meaſures A equally as E does C, *viz.* by the ſame number F; *b* alſo D equally meaſures C, as E does B, *viz.* by the ſame number G. *c* Therefore DF = A, and EG = B. *d* and conſequently A and B are plane numbers. But becauſe EF *c* = C *c* = DG, *e* ſhall D. E :: F. G. and alternately

a 35. 7.
b 21. 7.
c 9. *ax.* 7.
d 16. *def.* 7.
e 19. 7.

f 21. *def.* 7. nately D.F :: E.G. *f* Therefore the plane numbers A and B are alſo like. *Which was to be dem.*

PROP. XXI.

A, 16. C, 24. D, 36. B, 54.
E, 4. F, 6. G, 9.
H, 2. P, 2. M, 4. K, 3. L, 3. N, 6.

If between two numbers A, B there fall two mean proportional numbers C, D; thoſe numbers A, B are like ſolid numbers.

a 2. 8. *a* Take E, F, G, the leaſt ÷ in the proportion
b 10. 8. of A to C. *b* then E and G are like plane numbers: let the ſides of this be H and P, and of that
c 21. *def.* 7 K and L. *c* therefore H.K :: P.L :: *d* E. F. But E,
d *cor.* 18. 8. F, G, do *e* equally meaſure A, C, D, *viz.* by the
e 21. 7. ſame number M. and likewiſe the ſaid numbers E, F, G, do equally meaſure the numbers C, B, D, *viz.* by the ſame number N. *f* Therefore A = EM
f 9. *ax.* 7. = HPM, *f* and B = GN = KLN; *g* and ſo A and
g 17. *def.* 7. B are ſolid numbers. But for that C *f* = FM, and
h 17. 7. D *f* = FN, therefore ſhall M. N *h* :: FM. FN
k 7. 5. :: C.D *l* :: E.F :: H.K :: P. L. *m* wherefore A and
l *conſtr.* B are like ſolid numbers. *Which was to be dem.*
m 21. *def.* 7.

Lemma.

AE, BF, CG, DH,
A, B, C, D,
E, F, G, H.

If proportional numbers A, B, C, D, meaſure proportional numbers AE, BF, CG, DH, by the numbers E, F, G, H, theſe numbers (E, F, G, H,) ſhall be proportional.

For being AEDH *a* = BFCG, *a* and AD = BC,
a 19. 7.
b 1. *ax.* 7. *b* thence will $\frac{AEDH}{AD} = \frac{BFCG}{BC}$ *c* that is, EH = FG.
c 9. *ax.* 7. *a* Therefore E.F :: G.H. *Which was to be dem.*

Coroll.

d 15. *def.* 7. Hence $\frac{Bq}{Aq} = \frac{B}{A} \times \frac{B}{A}$ *d* For 1. B :: B. Bq *d* and 1. A ::
e *lem. prec.* A. Aq. *e* therefore $\frac{1}{A}$. $\frac{B}{A}$ :: $\frac{B}{A}$. $\frac{Bq}{Aq}$ *d* therefore $\frac{Bq}{Aq} = \frac{B}{A} \times \frac{B}{A}$

In like manner $\frac{B}{Ac} \times \frac{B}{Ac} = \frac{Bc}{Acc}$, and ſo of the reſt.

PROP.

PROP. XXII.

Aq, B, C. 4. 8. 16. *If three numbers Aq, B, C be continually proportional, and the first Aq a ſquare, the third C ſhall alſo be a ſquare.*

For becauſe Aq C *a* = Bq, *b* thence is $C = \frac{Bq}{Aq}$ $c = Q.\frac{B}{A}$; But it is plain that $\frac{B}{A}$ is a number, *d* becauſe $\frac{Bq}{Aq}$ or C is a number. Theref. if three, &c.

a 20. 7. b *ax.* 7. c *cor. of the lem. prec.* d *hyp. and* 14. 8.

PROP. XXIII.

Ac, B, C, D. 8, 12, 18, 27. *If four numbers Ac, B, C, D, be continually proportional, and the firſt of them Ac a cube, the fourth alſo D ſhall be a cube.*

For becauſe Ac D *a* = BC, *b* therefore $D = \frac{BC}{Ac}$ $c = \frac{B}{Ac} \times C$; that is (becauſe Ac C = *d* Bq, and *b* thence $C = \frac{Bq}{Ac}$) $D = \frac{B}{Ac} \times \frac{Bq}{Ac} c = \frac{Bc}{Acc} c = C \times \frac{B}{Ac}$.

a 19. 7. b 7. *ax.* 7. c *cor. of the prec. lem.* d 20. 7. e 15. 8.

But it is evident *c* that $\frac{B}{Ac}$ is a number, becauſe $\frac{Bc}{Acc}$ or D is ſuppoſed a number. Therefore if four numbers, &c.

PROP. XXIV.

A, 16. 24. B, 36. C, 4. 6. D, 9. *If two numbers A, B, be in the ſame proportion one to another, that a ſquare number C is to a ſquare number D, and the firſt A be a ſquare number, the ſecond alſo B ſhall be a ſquare number.*

Between C and D being ſquare numbers, * and ſo between A and B having the ſame proportion, *a* falls one mean proportional. Therefore *b* being

* 8. 8. a 11. 8. b *hyp.*

c 22. 8. A is a ſquare number, *c* B alſo ſhall be a ſquare number. *Which was to be dem.*

Coroll.

1. Hence, If there be two like numbers AB, CD (A. B :: C. D) and the firſt AB be a ſquare, the ſecond alſo CD ſhall be a ſquare.

*12. & 18 8. * For AB. CD :: Aq. Cq.

2. From hence it appears, that the proportion of any ſquare number to any other not ſquare, cannot poſſibly be declared in two ſquare numbers. Whence it cannot be Q. Q :: 1. 2. nor 1. 5 :: Q. Q, *&c.*

PROP. XXV.

C, 64. 96. 144. D, 216. A, 8. 12. 18. B, 27. *If two numbers A, B, be in the ſame proportion one to another, that a cube number C is to a cube number D, the firſt of them A being a cube number; the ſecond B ſhall likewiſe be a cube number.*

a 12. 8. b 8. 8. c *hyp.* d 23. 8. *a* Between the cube numbers C and D, *b* and ſo between A and B having the ſame proportion, fall two mean proportionals; therefore *c* becauſe A is a cube, *d* ſhall B be a cube alſo. *Which was to be dem.*

Coroll.

1. Hence, If there be two numbers ABC, DEF (A. B :: D. E, and B. C :: E. F;) and the firſt ABC be a cube, the ſecond DEF ſhall be a cube alſo.

*12, & 19, 8. * For ABC. DEF :: Ac. Dc.

1. It is perſpicuous from hence, that the proportion of any cube number to any other number not a cube cannot be found in two cube numbers.

PROP.

PROP. XXVI.

A,20. C,30. B,45. D,4. E,6. F,9.

Like plane numbers A, B, are in the ſame proportion one to another, that a ſquare number is in to a ſquare number.

Between A and B *a* falls one mean proportional number C; *b* take three numbers D,E,F, the leaſt ∴ in the proportion of A to C. the extremes D, F, *b* ſhall be ſquare numbers. But of equality A. B *c* :: D.F. therefore A. B :: Q.Q. *Which was to be dem.*

a 18. 8.
b 2. 8.
c 14. 7.

PROP. XXVII.

A,16. C,24. D,36. B,54. E,8. F,12. G,18. H,27.

Like ſolid numbers A, B, are in the ſame proportion one to another, that a cube number is in to a cube number.

a Between A and B fall two mean proportional numbers, namely C and D; *b* take four numbers E, F, G, H the leaſt ∴ in the proportion of A to C; *b* the extremes E, H, are cube numbers. But A. B *c* :: E. H :: C. C. *Which was to be dem.*

a 19. 8.
b 2. 8.
c 14. 7.

Schol.

1. From hence is inferred, that no numbers in proportion ſuperparticular, ſuperbipartient, or double, or any other manifold proportion not denominated from a ſquare number, are like plane numbers. See *Clavius*.

2. Likewiſe, that neither any two prime numbers, nor any two numbers prime one to another, not being ſquares, can be like plane numbers.

The End of the eighth Book.

THE NINTH BOOK OF EUCLIDE's ELEMENTS.

PROP. I.

A, 6. B, 54.
Aq, 36. 108. AB, 324.

IF two like plane numbers A, B multiplying one another, produce a number AB, the number produced AB ſhall be a ſquare number.

a 17. 7. b 18. 8. c 8. 8. d 22. 8.

For A. B *a* :: Aq. AB; wherefore ſince one mean proportional *b* falls between A and B, *c* likewiſe one mean proportional number ſhall fall between Aq and AB: therefore being the firſt Aq is a ſquare number, *d* the third AB ſhall be a ſquare number too. *Which was to be demonſtrated.*

Or thus. Let ab, cd be like plane numbers; namely a. b :: c.d. *x* therefore ad=bc. and ſo likewiſe ab cd, or ad bc, = ad ad = Q: ad.

x 19. 7. y 1. *ax.* 7.

PROP. II.

A, 6. B, 54. Aq, 36. AB, 324.

If two numbers A, B, multiplying one another, produce a ſquare number AB, thoſe numbers A, B are like plane numbers.

a 17. 7. b 11. 8. c 8. 8. d 20. 8.

For A. B *a* :: Aq. AB; wherefore being between Aq, AB, *b* there falls one mean proportional number, *c* likewiſe one mean ſhall fall between A and B. *d* therefore A and B are like planes. *Which was to be dem.*

PROP.

PROP. III.

A, 2. Ac, 8. Acc, 64. *If a cube number Ac multiplying it ſelf produce a number Acc. the number produced Acc ſhall be a cube number.*

For 1. A *a* :: A. Aq *b* :: Aq. Ac. therefore between 1 and Ac fall two mean proportionals. But 1. Ac *a* :: Ac. Acc. *c* therefore between Ac and Acc, fall alſo two mean proportionals: and ſo by conſequence ſeeing Ac is a cube, *d* Acc ſhall be a cube alſo. *Which was to be dem.*

a 15. *def.* 7.
b 17. 7.
c 8. 8.
d 13. 8.

Or thus; aaa (Ac) multiplied into it ſelf makes aaaaaa (Acc;) this is a cube, whoſe ſide is aa.

PROP. IV.

Ac, 8. Bc, 27. Acc, 64. AcBc, 216. *If a cube number Ac multiplying a cube number Bc, produce a number AcBc, the produced number AcBc ſhall be a cube.*

For Ac. Bc *a* :: Acc. AcBc. But between Ac and Bc *b* two mean proportional numbers fall; *c* therefore there fall as many between Acc and AcBc. So that whereas Acc is a cube number, *d* AcBc ſhall be ſuch alſo. *Which was to be dem.*

a 17. 7.
b 12. 8.
c 8. 8.
d 23. 8.

Or thus. AcBc = aaabbb (ababab) = C: ab.

PROP. V.

Ac, 8. B, 27. Acc, 64. AcB, 216. *If a cube number Ac multiplying a number B produce a cube number AcB, the number multiplied B ſhall alſo be a cube.*

For Acc. AcB. *a* :: Ac. B. But between Acc and AcB *b* fall two mean proportionals; *c* therefore alſo as many ſhall fall between Ac and B. whence Ac being a cube number, *d* B ſhall be a cube number too. *Which was to be dem.*

a 17. 7.
b 12. 8.
c 8. 8.
d 23. 8.

PROP. VI.

A, 8. Aq, 64. Ac, 512. *If a number A multiplying it ſelf produce a cube Aq, that number A it ſelf is a cube.*

a *hyp.* For becauſe Aq *a* is a cube, and AqA (Ac) *b* alſo a cube; therefore *c* ſhall A be a cube. *Which was to be dem.*

b 19. *def.* 7.
c 5. 9.

PROP. VII.

A, 6. B, 11. AB, 66. D, 2. E, 3. *If a compoſed number A multiplying any number B, produce a number AB, the number produced AB ſhall be a ſolid number.*

Being A is a compoſed number, *a* ſome other number D meaſures it, conceive by E. *b* therefore A = DE: *c* whence DEB = AB is a ſolid number. *Which was to be dem.*

a 13. *def.* 7.
b 9. *def.* 7.
c 17. *def.* 7.

PROP. VIII.

1. a, 3. a^2, 9 a^3, 27. a^4, 81. a^5, 243. a^5, 729.

If from a unit there be numbers continually proportional how many ſoever (1. a, a^2, a^3, a^4, &c.) *the third number from a unit* a^2 *is a ſquare number; and ſo are all forward, leaving one between* (a^4, a^6, a^8, &c.) *But the fourth* a^3 *is a cube number; and ſo are all forward, leaving two between* (a^6, a^9, &c.) *The ſeventh alſo* a^6, *is both a cube number and a ſquare; and ſo are all forward, leaving five between* (a^{12}, a^{18}, &c.)

For 1. a^2 = Q. a. and a^4 = aaaa = Q. aa. and a^6 = aaaaaa = Q. aaa, &c.

2. a^3 = aaa = C. a. and a^6 = aaaaaa = C. aa and aaaaaaaaa = C. aaa, &c.

3. a^6 = aaaaaa = C. aa = Q. aaa. therefore, &c.

a *hyp.* Or according to *Euclide*; Becauſe 1. a :: a. a^2.
b 20. 7. *b* ſhall a^2 = Q: a. therefore ſeeing a^2, a^3, a^4, are ::,
c 12. 8. *c* the third a^4 ſhall be a ſquare number; and ſo likewiſe a^7, a^8, &c. Alſo becauſe 1. a *a* :: a^2. a^1.
d 23. 8. therefore ſhall a^3 *b* = a^2 × a = C: a. *d* therefore the fourth from a^3, namely a^6, ſhall likewiſe be a cube, &c. and conſequently a^6 is both a cube and a ſquare number, &c.

PROP.

PROP. IX.

1.a.4.a^2,16.a^3,64.a^4,256,&c. 1.a.8.a^2,64.a^3,512.a^4,4096. *If from a unit there be numbers how many ſoever continually proportional* (1. a, a^2, a^3, &c.) *and the number following the unit* (a) *be a ſquare; then all the reſt*, a^2, a^3, a^4, &c. *ſhall be ſquares too. But if the number next the unit* (a) *be a cube, then all the following numbers* a^2, a^3, a^4, &c. *ſhall be cube numbers.*

1. *Hyp.* For a^2, a^4, a^6, &c. are ſquare numbers by the prec. Prop. alſo being a is taken to be a ſquare, *a* therefore the third a^3 ſhall be a ſquare, a 22. 8. and likewiſe a^5, a^7, &c. and ſo all.

2. *Hyp.* a is taken to be a cube, *b* therefore a^4, b 23. 8. a^7, a^{10} are cubes: but by the prec. a^3, a^6, a^9, c 20. 7. &c. are cubes: laſtly, becauſe 1.a :: a. aa. *c* there- d 3. 9. fore ſhall a^2 = Q: a. but a cube multiplied into c 23. 8. it ſelf *d* produces a cube; therefore a^2 is a cube, *e* and conſequently the fourth from it a^5, and in like manner a^8, a^{11}, &c. are cubes. therefore all. *Which was to be dem.*

Peradventure more clearly thus. let b be the ſide of the ſquare number a, and ſo the ſeries a, a^2, a^3, a^4, &c. will be otherwiſe expreſſed, thus, bb, b^4, b^6, b^8, &c. It is evident that all theſe numbers are ſquares, and may be thus expreſſed, Q: b, Q: bb: Q. bbb, Q: bbbb, &c.

In like manner, if b be the ſide of the cube a, the ſeries may be expreſſed thus, b^3, b^6, b^9, b^{12}. &c. or C: b, C: b^2, C: b^3, C: b^4, &c.

PROP. X.

1, a, a^2, a^3, a^4, a^5, a^6, 1, 2, 4, 8, 16, 32, 64, *If from a unit there be numbers how many ſoever continually proportional* (1, a, a^2, a^3, &c.) *and the number next the nnit* (a) *be not a ſquare number; then is none of the reſt following a ſquare number, excepting* a^2 *the third from the unit, and ſo all forward, leaving one be-*

tween (a^4, a^6, a^8, &c.) *But if that* (a) *which is next after the unit, be not a cube number, neither is any other of the following numbers a cube, ſaving* a; *the fourth from the unit, and ſo of all forward, leaving two between,* a^6, a^9, a^{12}, &c.

1. *Hyp.* For if it be poſſible, let a^5 be a ſquare number; therefore becauſe a. a^2 :: a^4, a^5, and by inverſion a^5. a4 :: a^2. a; and alſo a^5 and a4 are *b* ſquare numbers, and the firſt a^2 a ſquare, *c* therefore a ſhall be likewiſe a ſquare; *contrary to the Hypotheſis.*

a *hyp.* b *ſuppoſ.* & 8 9. c 24. 8.

2. *Hyp.* If it may be, let a4 be a cube; being *d* of equality a4. a^6 :: a. a^3 and inverſely a^6. a4 :: a3. a; and alſo being a^6 and a4 are cubes, and the firſt a3 a cube; *e* therefore a ſhall be a cube alſo; *againſt the Hyp.*

d 24. 7. e 25. 8.

PROP. XI.

1. a, a^2, a3, a4, a5, a6. 1, 3, 9, 27, 81, 243, 729. *If there be numbers how many ſoever in continual proportion from a unit* (1, a, a2, a3, &c.) *the leſs meaſureth the greater by ſome one of them that are amongſt the proportional numbers.*

Becauſe 1. a :: a. aa. *a* therefore $\frac{aa}{a} = a = \frac{aaa}{aa}$.

a 5. *ax.* 7. & 20. *def.* 7

Alſo becauſe 1. aa *b* :: a. aaa. *a* therefore $\frac{aaa}{a} = aa =$ $\frac{a4}{aa} = \frac{a5}{a3}$, &c. Laſtly becauſe 1. a3 *b* :: a. a4. therefore $\frac{a4}{a} = a3 = \frac{a6}{a3}$, &c.

b 14. 7.

Coroll.

Hence, If a number that meaſures any one of proportional numbers, be not one of the ſaid numbers, neither ſhall the number by which it meaſures the ſaid proportional numbers, be one of them.

PROP.

PROP. XII.

1, a, a2, a3, a4. 1,6,36,216,1296. B. 3.

If there be numbers how many ſoever in continual proportion from a unit) (1, a, a2, a3, a4.) *whatſoever prime numbers B meaſure the laſt* a4, *the ſame* (B) *ſhall alſo meaſure the number* (a) *which follows next after the unit.*

If you ſay B does not meaſure a, *a* then B is prime to a; *b* and alſo B is prime to a2; *c* and ſo conſequently to a4, which is ſuppoſed to meaſure. *Which is abſurd.*

a 31. 7. b 27. 7. c 26. 7.

Coroll.

1. Therefore every prime number that meaſures the laſt, does alſo meaſure all thoſe other numbers that precede the laſt.

2. If any number not meaſuring that next to the unit, does yet meaſure the laſt, it is a compoſed number.

3. If the number next to the unit be a prime, no other prime number ſhall meaſure the laſt.

PROP. XIII.

1, a. a2, a3, a4, 1, 5, 25, 125, 625. H--G--F--E--

If from a unit be numbers in continual proportion how many ſoever (a, a2, a3. &c.) *and that after the unit* (a) *a prime; then ſhall no other meaſure the greateſt number, but thoſe which are amongſt the ſaid proportional numbers.*

If it be poſſible, let ſome other E meaſure a4, viz. by F. *a* then F ſhall be ſome other beſide a, a2, a3. But becauſe E meaſuring a4, does not meaſure a, *b* therefore E ſhall be a compoſed number, *c* and ſome prime number meaſure it, *d* which does conſequently meaſure a4. *e* and ſo is no other than a, therefore a meaſures E. So alſo may F be ſhewn to be a compoſed number, meaſuring a4,

a *cor.* 11. 9. b 1. *cor.* 12. 9. c 33. 7. d 11. *ax.* 7. e 3. *cor.* 12.

measuring a4, and so that a measures F. There-
f 9. *ax.* 7. fore seeing EF *f* = a4 = a × a3, *g* shall a. E :: F.
g 19. 7. a3. Consequently, whereas a measures E, *b* like-
h 20. *def.* 7. wise F shall equally measure a3, *viz.* by the
k *cor.* 11. 9. same number G. *k* Nor shall G be a. or a2. there-
fore, as before, G is a composed number, and a measures it. Wherefore being that FG *f* = a3 = a2 × a *g* shall a. F :: G a2. and so because A measures F, *h* G shall equally measure a2. *viz.* by the same number H, *k* which is not a. There-
l 23. *def.* 7. fore being GH = a2 = aa, *l* thence H. a :: a.
m 20. *def.* 7. G. and because a measures G (as before) *m* H also shall measure a, which is a prime number. *Which is impossible.*

PROP. XIV.

A, 30. B, 2. C, 3. D, 5. E--F---

If certain prime numbers B, C, D, do measure the least number A, no other prime number E shall measure the same, besides those that measured it at first.

a 9. *ax.* 9. If it is possible, let $\frac{A}{E}$ be = F. *a* then A = EF.

b 32. 7. *b* therefore every of the prime numbers B, C, D, measures one of those E, F, not E, which is taken to be a prime; therefore F which is less than it self A; *contrary to the Hyp.*

PROP. XV.

A, 9. B, 12. C, 16. D, 3. E, 4.

If three numbers continually proportional A, B, C, be the least of all that have the same proportion with them; any two of them added together shall be prime to the third.

a 35. 7. *a* Take D and E the least in the proportion of
b 2. 8. A to B; *b* then A = Dq, & *b* C = Eq, *b* and B = DE.

But

But becauſe D *c* is prime to E, *d* therefore ſhall D + E be prime to both D and E. * therefore D x D + E *e* = Dq + DE (*f* A+B) is prime to E, and ſo to C or Eq. *Which was to be dem.*

g In like manner DE + Eq (B+C) is prime to D, and conſequently to A = Dq. *Which was to be dem.*

Laſtly, becauſe B *h* is prime to D+E, it ſhall alſo be prime to the ſquare of it *k* Dq + 2 DE + Eq (A + 2 B + C;) *l* wherefore the ſaid B ſhall be prime to A + B + C, *l* and ſo likewiſe to A + C. *Which was to be dem.*

c 2. 47.
d 30. 7.
* 26. 7.
e 3. 2.
f *before*
g 27. 7.
h 26. 7.
k 4. 2.
l 30. 7.

PROP. XVI.

A, 3. B, 5. C--- *If two numbers A, B, be prime to one another, the ſecond B ſhall not be to any other C, as the firſt A is to the ſecond B.*

If you affirm A. B :: B. C. then whereas A and B *a* are the leaſt in their proportion, A *b* ſhall meaſure B as many times as B does C; but A *c* meaſures itſelf alſo; therefore A and B are not prime to one another. *againſt the Hyp.*

a 23. 7.
b 21. 7.
c 6. *ax.* 7.

PROP. XVII.

A, 8. B, 12. C, 18. D, 27. E--- *If there be numbers how many ſoever in continual proportion A, B, C, D, and the extremes of them A, D be prime one to another, the laſt D ſhall no tbe to any other number E, as the firſt A is to the ſecond B.*

Suppoſe A. B :: D. E. then alternately A. D :: B. E. therefore ſeeing A and B are the leaſt in their proportion, A *b* ſhall meaſure B, *c* and B likewiſe C, and C the following number D, *d* and ſo A ſhall meaſure the ſaid number D. Wherefore A and D are

a 23. 7.
b 21. 7.
c 20. *def.* 7.
d 11. *ax.* 7.

are not prime to one another, *contrary to the Hypotheſis.*

PROP. XVIII.

A, 4. B, 6. C, 9. Bq, 36. *Two numbers being given A, B, to conſider if there may be a third number found proportional to them C.*

a 9. *ax.* 7. If A meaſure Bq by any number C, *a* then
b 20. 7. AC = Bq. from whence *b* it is manifeſt that A. B :: B. C. *Which was to be dem.*

A, 6. B, 4. Bq, 16. But if A do not meaſure Bq, there will not be any third proportional. For ſuppoſe A. B :: B. C. *a* then
c 7. *ax.* 7. AC = Bq. *c* and conſequently $\frac{Bq}{A}$ = C. namely A meaſures Bq. *Which is againſt the Hypoth.*

PROP. XIX.

A, 8. B, 12. C, 18. D, 27. BC, 216. *Three numbers being given A, B, C, to conſider if a fourth proportional to them D may be found.*

a 9. *ax.* 7. If A meaſures BC by any number D, *a* then
b *ax.* 19. 7. AD = BC; *b* therefore it appears that A. B :: C. D. which was required.

But if A do not meaſure BC, then there can no fourth proportional be found; which may be ſhewn as in the preceding Prop.

PROP. XX.

A, 2. B, 3. C, 5. D, 30. G---- *More prime numbers may be given than any multitude whatſoever of prime numbers A, B, C, propounded.*

a 38. 7. *a* Let D be the leaſt which A, B, C, meaſure; If
b 33. 7. D + 1 be a prime, the caſe is plain; if compoſed, *b* then ſome prime number, conceive G, meaſures

ures D + 1; which is none of the three A, B, C; For if it be, seeing it *c* measures the whole D − 1, *d* and the part taken away D, *e* it shall also measure the remaining unit. *Which is absurd.* Therefore the propounded number of prime numbers is increased by D + 1; or at least by G.

c *suppos.* d *constr.* e 12. *ax.* 7.

PROP. XXI.

5 5 3 3 2 2
A.....E.....B...F...C..G..D 20.

If even numbers, how many soever, AB, BC, CD, be added together, the whole AD shall be even.

a Take EB = ½ AB, and FC = ½ BC, and GD = ½ CD. *b* it is plain that EB + FC + GD = ½ AD. *c* therefore AD is an even number. *Which was to be dem.*

a 6. *def.* 7. b 12. 7. c 6. *def.* 7.

PROP. XXII.

A.....F.B.....G.C....H.D..L.E 22
9 7 5 3

If odd numbers, how many soever, AB, BC, CD, DE, be added together, and their multitudes even, the whole also AE shall be even.

A unit being taken from each odd number, there will *a* remain AF, BG, CH, DL, even numbers, *b* and thence the number compounded of them will be even, add to them the *c* even number made of the remaining units, and the *d* whole AE will thereby be even. *Which was to be dem.*

a 7. *def.* 7. b 21. 9. c *hyp.* d 21. 9.

PROP.

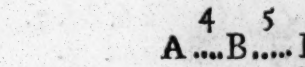
PROP. XXIII.

If odd numbers, how many ſoever, AB, BC, CD, be added together, and the multitude of them be odd, the whole AD ſhall be odd.

For CD one of the odd numbers being taken away, the number compounded of the others AC *a* is even. Whereto add CD — 1, *b* the whole AE is alſo even; wherefore the unit being reſtored, the whole AD *c* will be odd. *Which was to be dem.*

a 22. 9.
b 21. 9.
c 7. *def.* 7.

PROP. XXIV.



If an even number AB be taken away from an even number AC, that which remains BC ſhall be even.

For if BD (BC — 1) be odd, *a* BC (BD + 1) will be even. *Which was to be dem.* But if you ſay BD is even, becauſe AB *b* is even, *c* thence AD will be ſo; *a* and conſequently AC (AD+1) will be odd, *contrary to the Hypoth.* therefore BC is even. *Which was to be dem.*

a 7. *def.* 7.
b *hyp.*
c 21. 9.

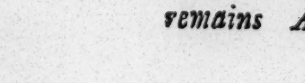
PROP. XXV.

If from an even number AB, an odd number AC be taken away, the remaining number CB ſhall be odd.

For AC—1 (AD) *a* is even. *b* therefore DB is even; *c* and conſequently CB (DB — 1) is odd. *Which was to be dem.*

a 7. *def.* 7.
b 24. 9.
c 7. *def.* 7.

PROP. XXVI.



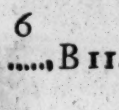
If from an odd number AB be taken away an odd number CB, that which remains AC ſhall be even.

For

For AB — 1 (AD) and CB — 1 (CD) *a* are even; *b* therefore AD — CD (AC) is even. *Which was to be done.*

a 7. *def.* 7.
b 24. 9.

PROP. XXVII.



If from an odd number AB be taken away an even number CB, the reſidue AC ſhall be odd.

For AB — 1 (DB) *a* is even, and CB is ſuppoſed to be even; *b* therefore the reſidue CD is even: *c* therefore CD + 1 (CA) is odd. *Which was to be dem.*

a 7. *def.* 7.
b 24. 9.
c 7. *def.* 7

PROP. XXVIII.

A, 3.
B, 4.
AB, 12.

If an odd number A multiplying an even number B produce a number AB, the number produced AB ſhall be even.

For AB *a* is compounded of the odd number A taken as many times as a unit is contained in B an even number. *b* Therefore AB is an even number.

a *hyp. and* 15. *def.* 7.
b 21. 9.

Schol.

In like manner, if A be an even number, AB ſhall be an even number alſo.

PROP. XXIX.

A, 3.
B, 5.
AB, 15.

If an odd number A multiplying an odd number B, produce a number AB, the number produced AB ſhall be odd.

For AB *a* is compounded of the odd number B taken as often as a unit is included in A likewiſe an odd number. *b* Therefore AB is an odd number. *Which was to be dem.*

a 15. *def.* 7.
b 23. 9.

Schol.

Schol.

$\frac{B, 12}{A, 3}$ (C, 4. 1. *An odd number A measuring an even number B, measures the same by an even number C.*

a 9. *ax.* 7. For if C be affirmed to be odd, then because a
b 29. 9. B = AC. *b* therefore B shall be odd, *against the Hypotheſis.*

$\frac{B, 15}{A, 3}$ (C, 5. 2. *An odd number A measuring an odd number B, measures the same by an odd number C.*

a 28. 9. For if C be ſaid to be even, *a* then AC, or B will be even, *contrary to the Hypotheſis.*

$\frac{B, 16}{A, 3}$ (C, 5. 3. *Every number (A and C) that measures an odd number B, is itself an odd number.*

For if either A or C be affirmed to be even, B
a 28. 9. *a* ſhall be an even number, *againſt the Hypoth.*

PROP. XXX.

$\frac{B, 24}{A, 3}$ (C, 8. $\frac{D, 12}{A, 3}$ (E, 4.

If an odd number A measure an even number B, it ſhall alſo measure the half of it D.

a *hyp.* *a* Let $\frac{B}{A}$ be = C. *b* then C is an even number.
b 1. *Schol.* Therefore let E be = $\frac{1}{2}$ C, then B *c* = CA *d* = 2
29. 9. EA *e* = 2 D. *f* therefore EA = D; *g* and conſe-
c 9. *ax.* 7. quently $\frac{D}{A}$ = E. *Which was to be dem.*
d 1. 2.
e *hyp.*
f 7. *ax.* 1.

PROP. XXXI.

g 7. *ax.* 7. A, 5. B, 8. C, 16. D--- *If an odd number A be prime to any number B, it ſhall alſo be prime to the double thereof C.*

If it be poſſible, let ſome number D meaſure A
a 3. *ſchol.* and C, *a* then D meaſuring the odd number A
29. 9. ſhall be odd it ſelf, *b* and ſo ſhall meaſure B the
b 30. 9. half of the even number C. therefr. A and B are not prime one to another. *Which is againſt the Hyp.*

Coroll.

Coroll.

It follows from hence that an odd number which is prime to any number of double progreſſion, is alſo prime to all the numbers of that progreſſion.

PROP. XXXII.

1. A, 2. B, 4. C, 8. D, 16. *All numbers A, B, C, D, &c. in double progreſſion from the binarie are evenly even only.*

It is evident that all theſe numbers 1, A, B, C, D, *a* are even, and *b* ÷, namely in a double proportion, *c* and ſo every leſs meaſures the greater by ſome one of them. *d* Wherefore all are evenly even. But for that A is a prime number, *e* no number beſides theſe ſhall meaſure any of them. Therefore they are evenly even only. *Which was to be dem.*

a 6. *def.* 7.
b 20. *def.* 7.
c 12. 9.
d 8. *def.* 7.
e 13. 9.

PROP. XXXIII.

A, 30. B, 15. D--- E-- *If of a number A, the half B be odd, the ſame A is evenly odd only.*

Being an odd number B *a* meaſures A by two an even number, *b* therefore B is evenly odd. If you affirm it to be evenly even, *c* then ſome even number D meaſures it by an even number E. whence 2 B *d* = A, *d* = DE. *e* wherefore 2. E :: D. B. and therefore as 2 *f* meaſures the even number E, *g* ſo D an even number meaſures B an odd. *Which is impoſſible.*

a *hyp.*
b 9. *def.* 7.
c 8. *def.* 7.
d 9. *ax.* 7.
e 9. 7.
f 6. *def.* 9.
g 20. *def.* 7.

PROP. XXXIV.

A, 24. *If an even number A be neither doubled from two, nor have its half part odd, it is both evenly even and evenly odd.*

It is undoubtable, that A is evenly even, becauſe the half of it is not odd. But becauſe, if A be divided into two equal parts, and its half again into

7. *def.* 7. into two equal parts and ſo on, we ſhall at length light upon ſome *a* odd number (not upon the number two, becauſe A is not ſuppoſed to be doubled upward from two) which ſhall meaſure
b 1 *ſch.* 29. A by an even number, ſo *b* otherwiſe A it ſelf
9. ſhould be odd, *againſt the Hypoth.* Therefore A alſo evenly odd. *Which was to be dem.*

PROP. XXXV.

A.....8.
4 8
B....F.....G12.
C.....18.
9 6 4 8
D.....H.....L....K.....N27.

If there be numbers in continual proportion how many ſoever A, BG, C, DN, and the number FG be taken from the ſecond, and KN from the laſt, equal to the firſt A; as the exceſs of the ſecond BF is to the firſt A, ſo ſhall the exceſs of the laſt DK be to all the numbers that precede it, A, BG, C.

From DN take NL = BG, and NH = C.
a *hyp.* Becauſe DN. C (HN) *a* :: HN. BG (LN) *a* ::
b 17. 5. LN (BG.) A (KN.) *b* therefore by dividing
c 12. 5. each, ſhall DH. HN :: HL. LN :: LK. KN. *c*
d 3. *ax.* 1. wherefore DK C + BG + A :: LK (*d* BF.) KN (A.) *Which was to be dem.*

Coroll.

e 18. 5. Hence *e* by compounding. DN + BG + C. A + BG + C :: BG. A.

PROP. XXXVI.

1. A,2. B,4. C,8. D,16.
E,31. G,62. H,124. L,248. F,496.
M,31. N,465.
P---- Q----

If from a unit be taken how many numbers ſoever 1, *A, B, C, D, in double proportion continually, until the whole added together E be a prime number; and if this*

whole

hole E multiplying the laſt produce a number F, that hich is produced F ſhall be a perfect number.

Take as many numbers E, G, H, L, likewiſe in double proportion continually; then *a* of equality A. D :: E. L. *b* therefore AL = DE *c* = F. *d* whence $L = \frac{F}{2}$. Wherefore E, G, H, L, F, are :: in double proportion. Let G − E be = M, and F − E = N; *e* then M. E :: N. E + G + H + L. *f* But M = E. *g* therefore N = E + G + H + L. *b* therefore F = 1 + B + C + D + E + G + H + L = E + N. Moreover because D *k* meaſures DE (F) *l* therefore every one, 1, A, B, C, *m* meaſuring D, as *m* alſo E, G, H, L, does meaſure F. And further, no other number meaſures the ſaid F. For if there do, let it be P, which meaſures F by Q. *n* therefore PQ = F = D. F. *o* therefore E.Q :: P.D. therefore ſeeing A a prime number meaſures D, *p* and ſo no other P meaſures the ſame, *q* conſequently E does not meaſure Q. Wherefore E being ſuppoſed a prime number, *r* it ſhall be prime to Q. *ſ* wherefore E and Q are the leaſt in their proportion; *t* and ſo E meaſures P as many times as Q does D; *u* therefore Q is one of them A, B, C. Let it be B, ſeeing then of equality B. D :: E. H :: *x* and ſo BH = DE = F = PQ. *x* and ſo alſo Q. B :: H. P. *y* therefore H = P. therefore P is alſo one of them A, B, C, *&c. againſt the Hypotheſis.* Wherefore no other beſide the foreſaid numbers meaſures F, and *z* conſequently F is a perfect number. *Which was to be demonſtrated.*

a 14. 7.
b 19. 7.
c *hyp.*
d 7. *ax.* 7.
e 35. 9.
f 3. *ax.*
g 14. 5.
h 2. *ax.* 1.
k 7. *ax.* 7.
l 11. *ax.* 7.
m 11. 9.
n 9. *ax.* 7.
o 19. 7.
p 13. 9.
q 20. *def.* 7.
r 31. 7.
ſ 23. 7.
t 21. 7.
u 13. 7.
x 19. 7.
y 14. 5.
z 22. *def.* 7.

The End of the ninth Book.

THE TENTH BOOK OF EUCLIDE's ELEMENTS.

Definitions.

I. COmmenſurable magnitudes are thoſe, which are meaſured by one and the ſame meaſure.

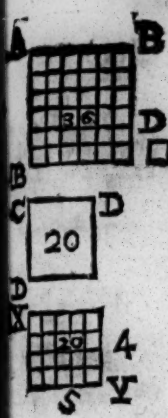
The note of commenſurability is ⊔, *as* A ⊔ B; *that is, the line* A *of* 8 *foot is commenſurable to the line* B *of* 13 *foot; becauſe* D *a line of one foot meaſures both* A *and* B. *Alſo* $\sqrt{18}$ ⊔ $\sqrt{50}$; *becauſe* $\sqrt{2}$ *meaſures both* $\sqrt{18}$ *and* $\sqrt{50}$. *For* $\sqrt{\frac{18}{2}} = \sqrt{9} = 3$. *and* $\sqrt{\frac{50}{2}} = \sqrt{25} = 5$. *wherefore* $\sqrt{18}.\sqrt{50} :: 3.5$.

II. Incommenſurable magnitudes are ſuch, of which no common meaſure can be found.

Incommenſurability is denoted by this mark ⊔; *as* $\sqrt{6}$ ⊔ $\sqrt{15}$ (5;) *that is,* $\sqrt{6}$ *is incommenſurable to the number* 5, *or to a magnitude deſigned by that number; becauſe there is no common meaſure of them, as ſhall appear hereafter.*

III. Right lines are commenſurable in power, when the ſame ſpace does meaſure their ſquares.

The



The mark of this commensurability is ⌐¬; *as AB* ⌐¬ *CD. i. e. the line AB of 6 foot is in power commensurable to the line CD, which is expressed by* √ 20. *because E the space of one foot square does as well measure ABq* (36) *as the rectangle XY* (20) *to which the square of the line CD* (√ 20) *is equal. The same note* ⌐¬ *sometimes signifies commensurable in power only.*

IV. Lines incommensurable in power are such; to whose squares no space can be found to be a common measure.

This incommensurability is denoted thus; 5 ⌐¬ *v* √ 8 *i. e. the numbers or lines* 5, *and v* √ 8 *are incommensurable in power, because their squares* 25 *and* √ 8 *are incommensurable.*

V. From which it is manifest, that to any right line given right lines infinite in multitude are both commensurable and incommensurable; some in length and power, others in power only. The right line given is called a Rational line.

The note of which is $\dot{\rho}$.

VI. And lines commensurable to this line, whether in length and power, or in power only, are also called Rational, $\dot{\rho}$.

VII. But such as are incommensurable to it, are called Irrational.

And denoted thus $\dot{\rho}$.

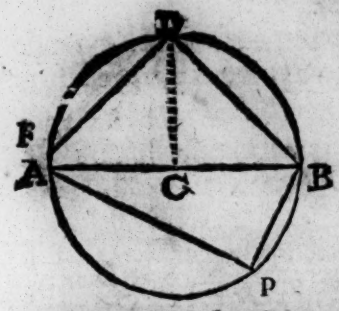
VIII. Also the square which is made of the said given right line is called Rational, $\dot{\rho}v$.

IX. And likewise such figures as are commensurable to it, are Rational, $\dot{\rho}\alpha$.

X. But such as are incommensurable, Irrational, $\dot{\rho}\alpha$.

XI. And those right lines also, which contain them in power, are Irrational $\dot{\rho}$.

Schol.



That the laſt ſeven definitions may be rendred more clear by an example, let there be a circle ADBP, whoſe ſemidiameter is CB, inſcribe therein the ſides of the ordinate figures, as of a Hexagone BP. of a triangle AP, of a ſquare BD, of a Pentagone FD. Therefore, if according to the 5. def. *the ſemidiameter CB be the Rational line given, expreſſed by the number* 2. *to which the other lines BP, AP, BD, FD, are to be compared, then* BP a $= BC = 2$. *wherefore BP is* $\dot{\rho}$ $\square$ *BC, according to the* 6. def. *Alſo* AP b $= \sqrt{12}$ (*for* ABq (16) $- BPq$ (4) $= 12$) *therefore AB* $\dot{\rho}$ $\square\!\!+$ *BC. likewiſe according to the* 6. def. *and* APq (12) *is* $\dot{\rho}r$ *by the* 9. def. *Moreover* BD b $= \sqrt{DCq + BCq} = \sqrt{8}$; *whence BD is* $\dot{\rho}$ $\square\!\!+$ *BC; and* BDq $\dot{\rho}r$. *Laſtly,* $FDq = 10 - \sqrt{10}$. (*as ſhall appear by the praxis to be delivered at the* 10. 13.) *ſhall be* $\dot{\rho}r$, *according to the* 10. def. *and conſequently* $FD = \sqrt{:10 - \sqrt{20}}$ *is* $\dot{\rho}$, *according to the* 11. def.

a cor. 15. 4.
b 47. 1.

A Poſtulate.

That any magnitude may be ſo often multiplied, till it exceed any magnitude whatſoever of the ſame kind.

Axiomes.

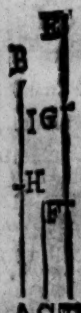
1. A magnitude meaſuring how many magnitudes ſoever, does alſo meaſure that which is compoſed of them.

2. A

2. A magnitude measuring any magnitude whatsoever, does likewise measure every magnitude which that measures.

3. A magnitude measuring a whole magnitude and a part of it taken away, does also measure the residue.

PROP. I.



Two unequal magnitudes AB, C, being given, if from the greater AB there be taken away more than half (AH) and from the residue (HB) be again taken away more than half (HI) and this be done continually, there shall at-length be left a certain magnitude IB, less than the less of the magnitudes first given C.

a Take C so often, till its multiple DE do somewhat exceed AB, and there be DE = FG = GE = C. Take from AB more than half HA, and from the remainder HB more than half HI, and so continually, till the parts AH, HI, IB, be equal in multitude to the parts DF, FG, GE. Now it is plain, that FE, which is not less than $\frac{1}{2}$ DE, is greater than HB, which is less than $\frac{1}{2}$ AB ⊐ DE. And in like manner GE, which is not less than $\frac{1}{2}$ FE, is greater than IB ⊐ $\frac{1}{2}$ HB. therefore C, or GE = IB. *Which was to be dem.*

a post. 10.

The same may also be demonstrated, if from AB the half AH be taken away, and again from the residue HB the half HI, and so forward.

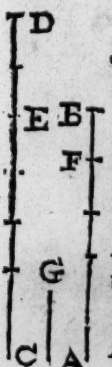
PROP. II.

Two unequal magnitudes being given (AB, CD) if the leſs AB, be continually taken from the greater CD, by an interchangeable ſubſtraction, and the reſidue do not meaſure the magnitude going before; then are the magnitudes given incommenſurable.

If it be poſſible, let ſome magnitude E be the common meaſure. Then becauſe AB taken from CD, as often as it can be, leaves a magnitude FD leſs than it ſelf, and FD taken from AB leaves GB, and ſo forward; *a* therefore at length ſome magnitude GB E ſhall be left. therefore E *b* meaſuring AB, *c* and ſo CF, *b* and the whole CD, *d* ſhall alſo meaſure the reſidue FD. *c* conſequently alſo AG; *d* wherefore it ſhall likewiſe meaſure the remainder GB, leſs than it ſelf. *Which is abſurd.*

a 1. 10.
b *hyp.*
c 2. *ax.* 10.
d 3. *ax.* 10.

PROP. III.



Two commenſurable magnitudes being given AB, CD, to find out their greateſt common meaſure FB.

Take AB from CD, and the reſidue ED from AB, and FB from ED, till FB meaſure ED (which will come to paſs at length, *a* becauſe by the Hyp. AB ⊓ CD) FB ſhall be the magnitude required.

For FB *b* meaſures ED, *c* and ſo alſo AF; but it meaſures it ſelf too, *d* therefore likewiſe AB, *c* and conſequently CE, *d* and ſo the whole CD. Wherefore FB is the common meaſure of AB, CD. If you affirm G to be a common meaſure greater than thar, then G meaſuring AB and CD, *e* meaſures alſo CE and *f* the remainder ED, *e* and ſo AF; and *f* conſequently the remainder FB, the greater the leſs. *Which is abſurd.*

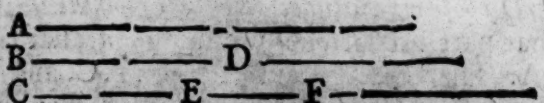
a 2. 10.
b *conſtr.*
c 2. *ax.* 10.
d 1. *ax.* 10.
e 2. *ax.* 10.
f 3. *ax.* 10.

Coroll.

Coroll.

Hence, A magnitude that measures two magnitudes, does also measure their greatest common measure.

PROP. IV.



Three commensurable magnitudes being given A, B, C, to find out their greatest common measure.

a Find out D the greatest common measure of any two A, B; *a* also E the greatest common measure of D and C. therefore E is the magnitude sought for. a 3. 10.

a For it is clear, E measuring D and C, *b* does measure the three A, B, C. Conceive another magnitude F greater than that to measure them; *c* then F measures D, *c* and consequently E the greatest common measure of D, and C, the greater the less. *Which is absurd.* b *constr.* & 2. *ax.* 10. c *cor.* 3. 10.

Coroll.

Hence also it appears, that if a magnitude measure three magnitudes, it shall likewise measure their greatest common measure.

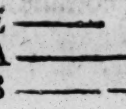
PROP. V.

A ——— D. 4.
C ——— F. 1.
B ——— E. 3.

Commensurable magnitudes A, B, have such proportion one to another, as number hath to number.

a C being found the greatest common measure of A, B; as often as C is contained in A and B, so often is 1 contained in the numbers D and E; *b* therefore C. A :: 1. D; wherefore inversely A. C :: D. 1. *b* but likewise C. B :: 1. E. *c* therefore of equality A. B :: D. E :: N. N. *W. W. to be dem.* a 3. 10. b 17. *def.* 7.

PROP. VI.



If two magnitudes A, B, have such proportion one to another as number C has to number D, those magnitudes A, B shall be commensurable.

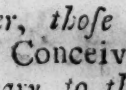
a *sch.* 10. 6. What part 1 is of the number C, *a* that let E
b *constr.* be of A. Therefore because E. A *b* :: 1. C. and A. B
c *hyp.* *c* :: C. D. *d* therefore of equality shall E. B :: 1. D.
d 21. 5. Wherefore seeing 1 *e* measures the number D, *f*
e 5. *ax.* 7. likewise E measures B; but it *g* also measures A.
f 20. *def.* 7. *h* therefore A ⊓ B. *Which was to be dem.*
g *constr.*
h 1. *def.* 10.

PROP. VII.

Incommensurable magnitudes A, B, have not that proportion one to another, which number has to number.

a 6. 10. If you affirm A. B :: N. N. *a* then A ⊓ B *against the Hypothesis.*

PROP. VIII.



If two magnitudes A, B, have not that proportion one to another, which number has to number, those magnitudes are incommensurable.

a 5. 10. Conceive A ⊓ B *a* then A. B :: N. N. *contrary to the Hypothesis.*

PROP. IX.

The squares described of right lines commensurable in length, have that proportion one to another, that a square number has to a square number. And squares, which have that proportion

portion one to another, that a ſquare number has to a ſquare number, ſhall alſo have their ſides commenſurable in length. But ſuch ſquares as are made of right lines incommenſurable in length, have not that proportion one to another, which a ſquare number has to a ſquare number. And ſquares which have not ſuch proportion one to another as a ſquare number has to a ſquare number, have not their ſides commenſurable in length.

1. *Hyp.* A. □ B. I ſay Aq. Bq :: Q. Q.

For *a* let A. B :: number E. number F. therefore $\frac{Aq}{Bq}$ (*b* $\frac{A}{B}$ twice) *c* = $\frac{E}{F}$ twice. *d* = $\frac{Eq}{Fq}$. *e* therefore Aq. Bq :: Eq. Fq :: Q. Q. *Which was to be dem.*

a 5. 10.
b 20. 6.
c *ſch.* 23. 5.
d 11. 8.
e 11. 5.

2. *Hyp.* Aq. Bq :: Eq. Fq :: Q. Q. I ſay A □ B. For $\frac{A}{B}$ twice (*f* $\frac{Aq}{Bq}$) *g* = $\frac{Eq}{Fq}$ *h* = $\frac{E}{F}$ twice. *i* therefore A. B :: E. F :: N. N. *k* wherefore A □ B. *Which was to be dem.*

f 20. 6.
g *hyp.*
h 11. 8.
i *ſch.* 23. 5.
k 6. 10.

3. *Hyp.* A □ C. I deny that Aq. Bq :: Q. Q. For ſuppoſe Aq. Bq :: Q. Q. then A □ B, as is ſhewn before, *againſt the Hyp.*

4. *Hyp.* Not Aq. Bq :: Q. Q. I ſay that A □ B, For conceive A □ B. then Aq. Bq :: Q. Q. as above, *againſt the Hyp.*

Coroll.

Lines □ are alſo ⏢. but not on the contrary. And lines □ are not therefore ⏢. but ⏢ are alſo □.

PROP.

PROP. X.

If four magnitudes be proportional (C. A :: B. D) and the first C be commensurable to the second A, the third B shall be commensurable to the fourth D. And if the first C be incommensurable to the second A, also the third B shall be incommensurable to the fourth D.

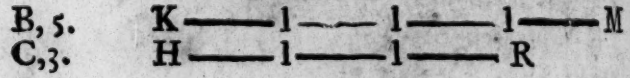
a 5. 10. If C ⊓ A, *a* then C. A :: N. N *b* :: B. D, *b*
b 6. 10. therefore B ⊓ D. But if ⊓ A, *c* then shall
c 7. 10. not C. A :: N. N :: B. D. *d* wherefore B ⊓ D.
d 8. 10. *Which was to be dem.*

Lemma 1.

To find out two plane numbers, not having the proportion which a square number hath to a square.

Any two plane numbers not like will satisfy this Lemma, as those numbers which have superparticular, superbipartient, or double proportion; or any two prime numbers. *See sch.* 27. 8.

Lemma 2.



To find out a line HR, to which a right line given KM hath the proportion of two numbers given B, C.

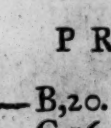
a *sch.* 10. 6. *a* Divide KM into as many equal parts as there are units in the number B. and let as
b 3. 1. many of these, as there are units in the number C, *b* make the right line HR, it is manifest that KM. HR :: B. C.

Lemma 3.

To find out a line D, to the square of which, the
a 12. *lem.* 10. *square of a right line given KM hath the proportion*
10. *of two numbers given B, C.*
b 13. 6. Allow B. C *a* :: KM. HR. and between KM
c 20. 6. and HR, *b* find a mean proportional D. There-
d *constr.* fore KMq. Dq *c* :: KM. HR *d* :: B. C.

PROP.

PROP. XI.



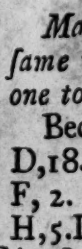
To find two right lines incommenſurable to a right line given A, one D in length only, the other E in power alſo.

1. Take the numbers B, C, *a* ſo that there be not B. C :: Q. Q. *b* and let B. C :: Aq. Dq. *c* it is plain that A ⊏ D. But Aq *d* ⊏ Dq. *Which was to be done.*

2. *d* Make A.E :: E.D. I ſay Aq ⊏ Eq. For A. D :: Aq. Eq. therefore ſince A ⊏ D, as before : *f* therefore Aq ⊏ Eq. *Which was to be done.*

a 1. *lem.* 10.
b 3. *lem.* 10.
c 9. 10.
d 13. 6.
e 20. 6.
f 10. 10.

PROP. XII.



Magnitudes (A, B) commenſurable to the ſame magnitude C, are alſo commenſurable one to the other.

Becauſe A ⊏ C, and C ⊏ B, *a* let A. D,18. E,8. C :: N. N :: D. E, and C. F, 2. G, 3. B :: N. N :: F. G. *b* take three numbers H, I, K, the leaſt in the proportions of D to E, and F to G. Now becauſe A.C *c* :: D.E *c* :: H.I, and C. B *c* :: F. G :: I. K. *d* therefore of equality A.B :: H.K :: N. N. *e* therefore A ⊏ B. *Which was to be dem.*

a 5. 10.
b 4. 8.
c *conſtr.*
d 22. 5.
e 6. 10.

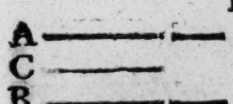
Schol.

Hence, Every right line commenſurable to a rational line is alſo it ſelf rational. And all right lines rational are commenſurable to one another, at leaſt in power. Alſo, every ſpace commenſurable to a rational ſpace is rational too : and all rational ſpaces are commenſurable one to another. But magnitudes whereof one is rational, the other irrational, are incommenſurable amongſt themſelves.

12. 10. *and def.* 6.
def. 9.
def. 7. & 10.

PROP.

PROP. XIII.



If there be two magnitudes A, B, and one of them A commenſurable to a third C, but the other B incommenſurable, thoſe magnitudes A, B are incommenſurable.

a *hyp.* Conceive B ⊏ A. then being C *a* ⊏ A, *b*
b 12. 10. therefore C ⊏ B, *againſt the Hyp.*

PROP. XIV.

If there be two magnitudes commenſurable A, B; and one of them A incommenſurable to any other magnitude C, the other alſo B ſhall be incommenſurable to the ſame C.

a *hyp.* Imagine B ⊏ C. then for that A *a* ⊏
b 12. 10. B, *b* therefore A ⊏ C, *againſt the Hyp.*

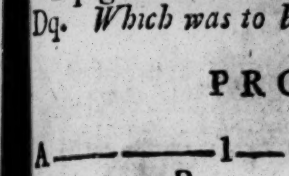
PROP. XV.

If four right lines be proportional (A. B :: C. D.) and the firſt A be in power more than the ſecond B by the ſquare of a right line commenſurable to it ſelf in length, then alſo the third C ſhall be more in power than the fourth D by the ſquare of a right line commenſurable to it ſelf in length. But if the firſt A be more in power than the ſecond B by the ſquare of a right line incommenſurable to it ſelf in length, then ſhall the third C be more in power than the fourth D by the ſquare of a right line incommenſurable to it ſelf in length.

a *hyp.* For becauſe A. B *a* :: C. D. *b* therefore Aq. Bq ::
b 22. 6. Cq. Dq. *c* therefore by diviſion Aq Bq. Bq :: Cq
c 17. 5. — Dq. Dq. *d* wherefore √ : Aq — Bq. B :: √ : Cq —
d 22. 6. Dq. D. *e* and ſo inverſely B. √ : Aq Bq :: D. √ : Cq.
e *cor.* 4. 5. Dq. *f* therefore of equality A. √ : Aq — Bq :: C.
f 22. 5. √ : Cq — Dq. conſequently if A ⊏, or ⊐ √ Aq. Bq,

-Bq. *g* then likewiſe C □, or □ √ : Cq — g 10. 10.
Dq. *Which was to be dem.*

PROP. XVI.



If two magnitudes commenſurable AB, BC, be compoſed, the whole magnitude AC ſhall be commenſurable to each of the parts AB, BC. And if the whole magnitude AC be commenſurable to either of the parts AB, or BC, thoſe two magnitudes given at firſt AB, BC, ſhall be commenſurable.

1. *Hyp. a* Let D be the common meaſure of AB, BC; *b* ſo D meaſures AC. and therefore AC □ AB, and BC. *Which was to be dem.*

a 3. 10.
b 1. *ax.* 10.
c 1. *def.* 10.

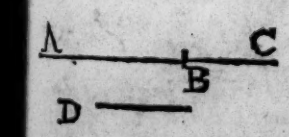
2. *Hyp. a* Let D be the common meaſure of AC, AB. *d* therefore D meaſures AC - AB (BC) and conſequently AB □ BC. *Which was to be dem.*

d 3. *ax.* 10.

Coroll.

Hence it follows, if a whole magnitude compoſed of two, be commenſurable to any one of them, the ſame ſhall be commenſurable to the other alſo.

PROP. XVII.



If two incommenſurable magnitudes AB, BC, be compoſed, the whole magnitude alſo AC ſhall be incommenſurable to either of the two parts AB, BC. And if the whole magnitude AC be incommenſurable to one of them AB, the magnitudes firſt given AB, BC, ſhall be incommenſurable.

1. *Hyp.* If it can be, let D be the common meaſure of AC, AB. *a* therefore D meaſures AC - AB (BC) *b* and therefore alſo AB □ BC, *againſt the Hypoth.*

a 3. *ax.* 10.
b 1. *def.* 10.

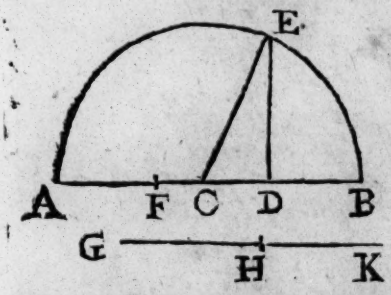
2. *Hyp.*

c 16. 10. 2. *Hyp.* Conceive AB ⊓ BC. *c* therefore AC ⊓ AB, *againſt the Hyp.*

Coroll.

Hence alſo, If one magnitude, compoſed of two, be incommenſurable to any one of them, the ſame alſo ſhall be incommenſurable to the other.

PROP. XVIII.



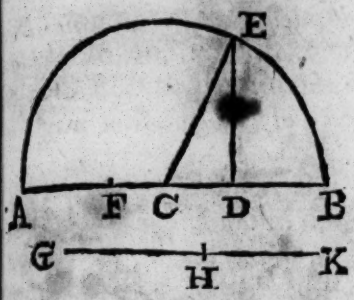
If there be two unequal right lines AB, GK and upon the greater AB a parallelogram ADB equal to the fourth part of a ſquare made of the leſs line GK, and wanting in figure by a ſquare, be applied, and divide the ſaid AB into parts commenſurable in length AD, DB; then ſhall the greater line AB be more in power than the leſs GK by the ſquare of a right line FD commenſurable in length to the greater. And if the greater AB be in power more than the leſs GK by the ſquare of the right line FD commenſurable unto it ſelf in length, and a parallelogram ADB equal to the fourth part of the ſquare made of the leſs line GK, and wanting in figure by a ſquare, be applied to the greater AB, then ſhall it divide the ſame into parts AD, DB commenſurable in length.

a 10. 1. b 28. 6. c 8. 2. d *conſtr.* & 4. 2. e 16. 10. g *cor.* 16. 10. k 12. 10. l 16. 10.

a Divide GK equally in H, and *b* make the rectangle ADB = GHq. Cut off AF = DB. then is ABq *c* = 4 ADB *d* (4 GHq or GKq) + FD. Now in the firſt place, if AD ⊓ DB. then ſhall AB *e* ⊓ BD *c* ⊓ 2 DB *f* (AF + DB, or AB—FD) *b* therefore AB ⊓ FD. *Which was to be dem.* But ſecondly, if AB ⊓ FD, *b* then ſhall AB □ AB — FD (2 DB) *k* therefore AB ⊓ DB. *l* wherefore AD ⊓ DB. *Which was to be dem.*

PROP.

PROP. XIX.



If there be two right lines unequal AB, GK, and to the greater AB be applied a parallelogram ADB equal to the fourth part of a ſquare made upon the leſs GK, and wanting in figure by a ſquare, and alſo thus applied divide the ſaid AB into parts AD, DB incommenſurable in length; the greater line AB ſhall be in power more than the leſs GK by the ſquare of the right line FD incommenſurable to the greater in length. And if the greater line AB be more in power than the leſs GK by the ſquare of a right line FD incommenſurable unto it ſelf in length, and if alſo upon the greater AB be applied a parallelogram ADB equal to the fourth part of the ſquare of the leſs GK and wanting in figure by a ſquare, then ſhall it divide the ſaid greater line AB into parts incommenſurable in length AD, DB.

Suppoſe all the ſame that was done and ſaid in the prec. Prop. Therefore firſt, If AD ▭ DB, *a* then ſhall AB ▭ DB. *b* Wherefore AB ▭ 2 DB (AB — FD) *c* therefore AB ▭ FD. *Which was to be dem.*

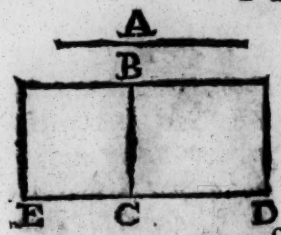
a 17. 10.
b 13. 10.
c *cor.* 17. 10.

Secondly, If AB ▭ FD, then AB ▭ AB—FD (2 DB;) *d* wherefore AB ▭ DB, *e* and conſequently AD ▭ DB. *Which was to be dem.*

d 13. 10.
e 17. 10.

PROP.

PROP. XX.

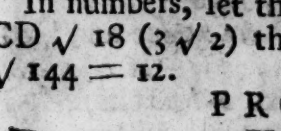


A rectangle BD comprehended under right lines BC, CD, rational and commensurable in length, according to one of the foresaid ways, is rational.

Let A be given ῥ, and
a 46. 1. *a* the square BE described upon BC. Because DC,
b 1. 6. CE (BC) *b* :: BD. BE. and DC *c* ▭ BC, *d*
c *hyp.* therefore shall the rectangle BD be ▭ square
d 10. 10. BE. wherefore seeing the square BE *e* ▭ Aq,
e *hyp. and* shall also BD be ▭ Aq. and so the rectangle
9 *def.* 10. BD is ῥν. *Which was to be dem.*
f 12. 12. Note, *There are three kinds of lines rational commensurable one to another. For either, of two lines rational commensurable in length one to the other, one is equal to the rational line propounded, or neither of them is equal to it, notwithstanding both of them are commensurable to it in length; or lastly, both of them are commensurable to the rational line given only in power. And these are the ways which the present Theorem speaks of.*

In numbers, let there be BC $\sqrt{8}$ ($2\sqrt{2}$) and CD $\sqrt{18}$ ($3\sqrt{2}$) then shall the rectangle BD = $\sqrt{144} = 12$.

PROP. XXI.



If a rational rectangle DB be applied to a rational line DC, it makes the breadth thereof CB rational, and commensurable in length to that line DC, whereto DB is applied.

a 1. 6. Let G be propounded ῥ, and the square DA
b *hyp.* described on BC. because BD. DA :: *a* BC. CA;
c *sch.* 12. 10. and BD, DA *b* are ῥα *c* and so ▭. *d* therefore
d 10. 10. BC ▭ CA. but CD (CA) is ῥ. *e* therefore BC
e *sch.* 12. 10. is ῥ. *Which was to be dem.*

In

In numbers, let there be rectangle DB. 12, and DC, $\sqrt{8}$. then shall CB, $\sqrt{18}$. but $\sqrt{18} = 3\sqrt{2}$. and $\sqrt{8} = 2 \times \sqrt{2}$.

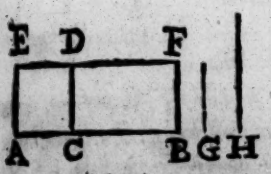
Lemma.

To find out two right lines rational commensurable only in power.

Let A be propounded $\dot{\rho}$. *a* Take B ⊔ A, *a* and C ⊔ B. *b* it is clear that B and C are the lines required.

a 11. 10.
b *sch.* 12. 10.

PROP. XXII.



A rectangle DB comprehended under right lines rational DC, CB, commensurable in power only, is irrational; and the right line H, which containeth that rectangle in power is irrational, and called a Medial line.

Let G be the propounded $\dot{\rho}$, and the square DA described on DC, and let Hq = DB. Because AC. CB *a* :: DA. DB. *b* and AC ⊔ CB, *c* shall be DA ⊔ DB (Hq,) *d* but Gq ⊔ DA. *e* therefore Hq ⊔ Gq. *f* wherefore H is $\dot{\rho}$. *Which was to be dem.* and let it be called a Medial line, because AC. H :: H. CB.

a 1. 6.
b *hyp.*
c 10. 10.
d *hyp. and* 9 3. *def.* 10.
e 13. 10.
f 11. 10.

In numbers, let there be DC, 3. and CB, $\sqrt{6}$. then shall the rectangle be DB (Hq) $\sqrt{54}$. wherefore H is $v\sqrt{54}$.

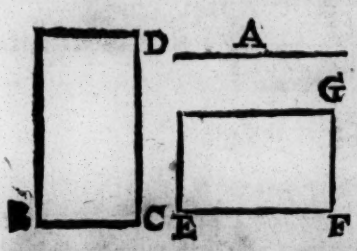
The note of a medial line is μ, of a medial rectangle $\mu\rho$, of more together $\mu\alpha$.

Schol.

Every rectangle that can be contained under two right lines rational commensurable only in power, is medial, altho' it be contained under two right lines irrational: and every medial rectangle may be contained under two right lines rational, commensurable only in power, as for example,

the √ 24 is μν, because it is contained under √ 3, and √ 8, which are ῥ, ⊐. altho' it may be contained under ν √ 6, and ν √ 96 irrationals; for √ 24 = ν √ 576 = ν √ 6 × ν √ 96.

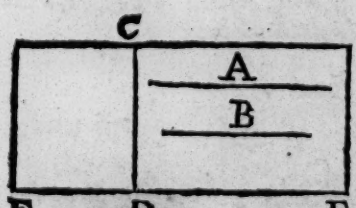
PROP. XXIII.



If the rectangle BD made of a medial line A, be applied on a rational line BC, it makes the breadth CD rational, and incommensurable in length to the line BC, whereunto the rectangle BD is applied.

a *sch.* 12. 10. Because A is μ, *a* therefore shall Aq be equal to some rectangle (EG) contained under EF and
b 1. *ax.* 1. FG ῥ ⊐. *b* therefore BD = EG. *c* whence BC.
c 14. 6. EF :: FG. CD. *d* therefore BCq. EFq :: FGq. CDq.
d 22. 6. But BCq and EFq *e* are ῥα, *f* and so ⊏. *g* therefore FGq ⊏ CDq. Wherefore being FG is ῥ,
e *hyp.* *h* therefore CD shall be ῥ. Moreover, because
f *sch.* 12. 10. EF. FG *k* :: EFq. EG (BD;) for that EF ⊏ FG,
g 10. 10. *e* shall EFq be ⊏ BD. But EFq *m* ⊏ CDq. *n*
h *sch.* 12. 10. therefore the rectangle BD ⊏ CDq. Whence being CDq. BD *o* :: CD. BC. *p* shall CD be ⊏ BC. therefore, *&c.*
k 1. 6.
l 10. 10.
m *sch.* 12. 10.
n 13. 10.
o 1. 6.
p 10. 10.

PROP. XXIV.



A right line B commensurable to a medial line A is also a medial line.

Upon CD ῥ *a* make the rectangle CE = Aq; *a* and the rectangle CF = Bq. Because Aq (CE) is μν, *b* and CD ῥ. *e* therefore shall the latitude DE be ῥ ⊐ CD. But for that CE. CF *d* :: ED. DF. and CE *e* ⊏ CF,

a 11. 6.
b *hyp.*
c 23. 10.
d 1. 6.
e *hyp.*

CF, *f* therefore ED ▭ DF. *g* therefore DF is *ῥ* ▭ CD. *h* whence the rectangle CF (Bq) is *μῥ*, and so B is *μ*. *Which was to be dem.*

f 10. 10. g 12. *and* 13. 10. h 22. 10.

Obs. *That the note* ⌜ *for the most part signifies commensurable in power only, as in this and the precedent demonstrations,* &c.

Coroll.

Hereby it is manifest that a space commensurable to a medial space, is also medial.

Lemma.

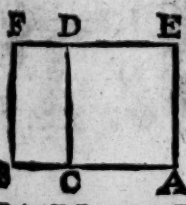
A ——— B ——— C ———

To find out two right lines medial A, B, commensurable in length, and also two, A, C, commensurable only in power.

a Let A be any *μ*, *b* take B ▭ A, and *c* C ⌜ A. *d* it appears to be done.

a *lem.* 22. 10. *and* 13. 6. b 2 *lem.* 10. 10. c 3 *lem.* 10. 10. d *constr.* & 24. 10.

PROP. XXV.

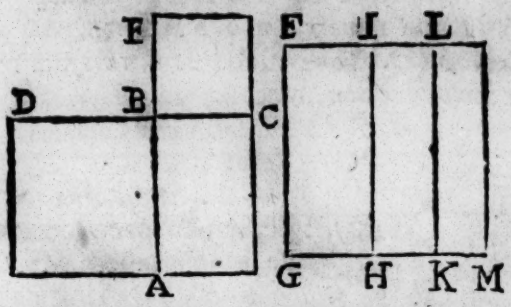


A rectangle DB contained under DC, CB medial right lines commensurable in length, is medial.

Upon DC describe the square DA. Being AC (DC.) CB *a* :: DA. DB. and DC ▭ CB; *b* shall DA ▭ DB. *c* therefore DB is *μῥ*. *Which was to be dem.*

a 1. 6. b 10. 10. c 24. 10.

PROP. XXVI.

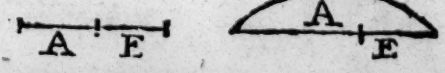


A rectangle AC comprehended under medial right lines AB, BC commensurable only in power, is either rational or medial.

a 46. 1. Upon the lines AB, BC, *a* describe the squares AD, CE; and upon FG $\dot{\rho}$ make the rectangles FH,
b *cor.* 16. 6. = AD, *b* and IK = AC. *b* and LM = CE.

The squares AD, CE, that is, the rectangles
c *hyp.* & FH, LM, *c* are $\mu\alpha$ and $\square$. therefore GH, KM,
24. 10. having the same proportion *d* are $\dot{\rho}$, *e* and $\square$.
d 23. 10. *f* therefore GH × KM is $\dot{\rho}\nu$. But because AD,
e 10. 10. AC, CE. that is, FH, IK, LM, *g* are $\div\div$; *b* and
f 20. 10. so GH, HK. KM also $\div\div$; *k* thence HKq = GH
g *sch.* 22. 6. × KM. *l* therefore HK is $\dot{\rho}$ or $\square$, or $\square$ IH
h 1. 6. (GF;) if $\square$, *m* then the rectangle IK or AC is
k 17. 6. $\dot{\rho}\nu$. but if $\square$, *n* then AC is $\mu\nu$. *Which was to*
l 12. 10. *be dem.*
m 20. 10.
n 22. 10.

Lemma.



If A and E be $\square$ *only, Then first shall Aq,*

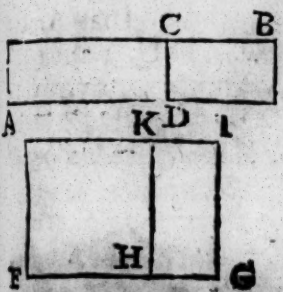
a *hyp. and* Eq, Aq + Eq, Aq − Eq *a* $\square$. *And secondly* Aq, Eq,
10. 10. Aq + Eq. Aq − Eq $\square$ AE and 2 E. For A. E *b* ::
b 1. 6. Aq. AE *b* :: AE. Eq. therefore seeing A *c* $\square$ E,
c *hyp.* *d* shall Aq $\square$ AE, *e* and 2 AE. also Eq *d* $\square$
d 10. 10. AE, *e* and 2 AE. wherefore because Aq + Eq $\square$
e 14. 10. Aq and Eq; and Aq − Eq $\square$ Aq and Eq. *f*
f 14. 10. therefore shall Aq + Eq. *f* and Aq − Eq be $\square$
AE, and 2 AE.

Hence

Hence also thirdly, Aq. Eq. Aq + Eq, Aq − Eq, ± 2 AE *g* □ Aq + Eq ± 2 AE; and Aq + Eq − 2 AE *g* and Aq + Eq + 2 AE □ Aq + Eq − 2 AE. *h* (Q. A − E.)

g 14. 16.
& 17. 10.
h cor. 7. 12.

PROP. XXVII.

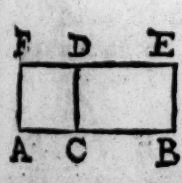


A medial rectangle AB exceedeth not a medial rectangle AC by a rational rectangle DB.

Upon EF ṗ, *a* make EG = AB, *a* and EH = AC. The rectangles AB, AC, *i. e.* EG, EH, *b* are μα; *c* therefore FG and FH are ṗ □ EF. Whence, if KG, *d i. e.* DB be ṗv. *e* then shall HG be □ HK; *f* wherefore HG □ FH. *g* and consequently FGq □ FHq. But FH is ṗ. *h* therefore is FG ṗ. but FG was ṗ before. *Which is contradictory.*

a cor. 16. 6.
b hyp.
c 23. 10.
d 3. ax. 1.
e 21. 10.
f 13. 10.
g lem. 16. 10.
b sch. 12. 10.

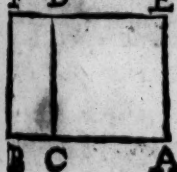
Schol.



1. *A rational rectangle* AE *exceeds a rational rectangle* AD *by a rational rectangle* CE.

For AE *a* □ AD. *b* therefore AE □ CE. *c* wherefore CE is ṗv. *Which*, &c.

a hyp.
b cor. 16. 10.
c sch. 12. 10.



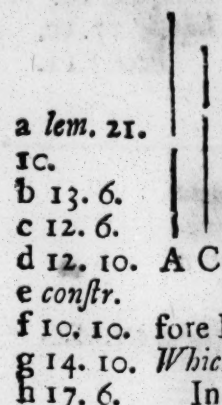
2. *A rational rectangle* AD *joyned with a rational rectangle* CF *makes a rational rectangle* AF.

For AD *a* □ AF. *b* wherefore AF □ AD and CF, *c* and so AF is ṗv. *Which was to be dem.*

a sch. 11. 10.
b 16. 11,
c sch. 12.

PROP. XXVIII.

To find out medial lines (C and D) which contain a rational rectangle CD.



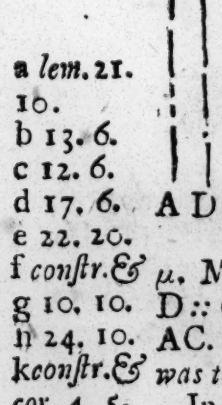
a Take A and B ρ ⊐. *b* make A.C :: C.B. *c* and A.B :: C. D. I say the thing required is done. For AB (Cq) *d* is μ_{ν}, *d* whence C is μ. but being that A. B :: C.D. *f* therefore C ⊐ D. *g* and consequently D is μ. Moreover by inversion A.C :: B.D. *i. e.* C.B :: B.D. *h* therefore Bq = CD. But Bq is ρ_{ν}. *b* therefore CD is ρ_{ν}. *Which was to be done.*

In numbers, let A be $\sqrt{2}$; and B $\sqrt{6}$. therefore C is $v\sqrt{12}$. make $\sqrt{2}$. $\sqrt{6}$ :: $v\sqrt{12}$. D. or $v\sqrt{4}$. $v\sqrt{36}$ :: $v\sqrt{12}$. D. then shall D be $v\sqrt{108}$. but $v\sqrt{12} \times v\sqrt{108} = v\sqrt{1296} = \sqrt{36}$. = 6. therefore CD is 6. likewise C. D :: 1. $\sqrt{3}$. wherefore C ⊐ D.

a *lem.* 21.
10.
b 13. 6.
c 12. 6.
d 12. 10.
e *constr.*
f 10. 10.
g 14. 10.
h 17. 6.
k *sch.* 12. 10.

PROP. XXIX.

To find out medial right lines commensurable in power only, D and E, containing a medial rectangle DE.



a Take A, B, C, ρ ⊐. make A.D *b* :: D. B. *c* and B. C :: D. E. I say the thing desired is performed.

For AB *d* = Dq. and AB *e* is μ_{ν}, therefore D is μ; and B *f* ⊐ C. *g* whence D ⊐ E. therefore *h* E is μ. Moreover B, C *f* :: D. E. and by inversion B. D :: C. E. *i. e.* D. A :: C. E. *l* therefore DE = AC. But AC *m* is μ_{ν}. therefore DE is μ_{ν}. *Which was to be done.*

In numbers, let A be 20. and B, $\sqrt{200}$, and C, $\sqrt{80}$. Therefore D is $\sqrt{\sqrt{80000}}$; and E $v\sqrt{12800}$. Therefore DE = $\sqrt{\sqrt{1024000000}}$ = 32000, and D.E :: $\sqrt{10}$, 2, wherefore D ⊐ E.

a *lem.* 21.
10.
b 13. 6.
c 12. 6.
d 17. 6.
e 22. 20.
f *constr.* & g 10. 10.
h 24. 10.
k *constr.* & *cor.* 4. 5.
l 16. 6.
m 22. 6.

Schol.

Schol.

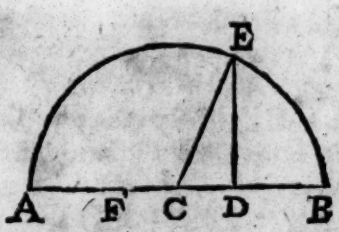
A,6. C,12.
B,4. D,8.
AB,24. CD,96.

To find out plane numbers, like or unlike.

Take any four numbers proportional A. B :: C. D. it is manifeſt that AB and CD are like plane numbers. And you may find out as many unlike plane numbers, as you pleaſe, by help of *Schol.* 27. 8.

A,6. C,5.
B,4. D,8.
AB,24. CD,40.

Lemma.



To find out two ſquare numbers (DEq and CDq) ſo that the number compoſed of them (CEq) be ſquare alſo.

Take AD, DB like plane numbers (of which let both be equal, or both odd) *viz.* AD, 24. and DB, 6. The total of theſe (AB) is 30; the difference (FD) 18 half of which (CD) is 9.

a Now the like plane numbers AD, DB, have one mean number proportional, namely DE. therefore it is evident that every of thoſe numbers CE, CD, DE, are rational, and by conſequence CEq (*b* CDq + DEq) is the ſquare number required. a 18. 8. b 47. 1.

Whereby it will be eaſy to find out two ſquare numbers, the exceſs of which is a ſquare or not a ſquare number, namely by the ſame conſtruction *c* ſhall CEq — CDq be = DEq. c 3. *ax.* 1.

But if AD, DB be plane numbers unlike, the

medial proportional line (DE) ſhall not be a rational number, and ſo neither ſhall the exceſs (DEq) of the ſquare numbers, CEq, CDq, be a ſquare number.

Lemma 2.

2. *To find out two ſuch ſquare numbers B, C, as the number compounded of them D is not ſquare. Alſo to divide a ſquare number A into two numbers B, C, not ſquares.*

A, 3. B, 9. C, 36. D, 45.

1. Take any ſquare number B, and let C be = 4 B, and D = B + C. I ſay the matter is done.

For B is Q. by the conſtr. likewiſe becauſe
a 24. 8. B. C :: 1. 4 :: Q. Q. *a* therefore C alſo ſhall be a ſquare number. But becauſe B + C. (D) C ::
b cor. 24. 8. 5. 4 :: not Q. Q. *b* therefore ſhall not D be a ſquare number. *Which was to be done.*

A, 36. B, 24. C, 12. D, 3. E, 2. F, 1.

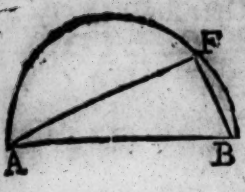
2. Let A be ſome ſquare number. Take D, E, F, plane numbers alike, and let D be = E + F. make D. E :: A. B. and D. F :: A. C. I ſay the thing required is done.

For becauſe D. E + F :: A. B + C. and D = E + F. *a* therefore ſhall A = B + C. Now ſup-
a 14. 5. b 21. *def.* 7. poſe B to be ſquare, *b* then A and B, *c* and con-
c 26. 8. ſequently D and E are like plane numbers. *Which is contrary to the Hyp.*

The ſame abſurdity will follow if C be ſuppoſed a ſquare number. Therefore, *&c.*

PROP.

PROP. XXX.



To find out two ſuch rational right lines AB, AF, commenſurable only in power, as the greater AB ſhall be in power more than the leſs AF by the ſquare of a right line BF commenſurable in length to the greater.

Let AB be the line given ῥ. *a* Take the ſquare numbers CD, CE, ſo that CD—CE (ED) be not Q. *b* and let there be CD. ED :: ABq. AFq. In a circle deſcribed upon the diameter AB *c* draw AF, and alſo BF. Then I ſay AB, AF, are the lines required.

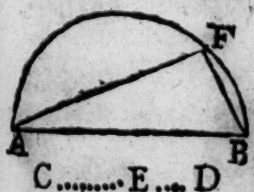
a 1. *lem.* 29. 10.
b 3. *lem.*
c 1. 4.
d *conſtr.*

For ABq. AFq *d* :: CD. ED. *e* therefore ABq ⊓ AFq. but AB is ῥ. *f* therefore AF is alſo ῥ. But becauſe CD is Q: and ED not Q: *g* therefore ſhall AB be ⊓ AF. Moreover by reaſon of the *h* right angle AFB, is ABq *k* = AFq + BFq; therefore ſeeing ABq. AFq :: CD. ED. by converſion of proportion ſhall ABq. BFq :: CD. CE :: Q. Q. *l* therefore AB ⊓ BF. *Which was to be done.*

e 6. 10.
f *ſch.* 12. 10.
g 9. 10.
h 3. 1. 3.
k 47. 1.
l 9. 10.

In numbers, let there be AB, 6; CD, 9; CE, 4; wherefore ED, 5. Make 9. 5 :: 36. (Q: 6) AFq. then AFq ſhall be 20. and conſequently AF √ 20. therefore BFq = 36 — 20 = 16. wherefore BF is 4.

PROP. XXXI.



To find out two rational lines AB, AF commenſurable only in power, ſo that the greater AB ſhall be in power more than the leſs AF by the ſquare of a right line BF incommenſurable in length to the greater.

Let AB be the line given ῥ. *a* Take the ſquare numbers CE, ED, ſo that CD = CE + ED be not Q. and in the reſt follow the conſtruction of the preced. Prop. I ſay then the thing required is done.

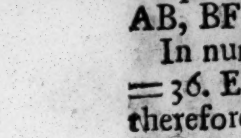
a 2. *lem.* 19. 10.

For

b 9. 10. For, as above, AB, AF, are $\dot{\rho}$ ⌐⌐. alſo ABq. BFq :: CD. ED. therefore being CD is not Q. AB, BF *b* ſhall be ⌐⌐. *Which was to be done.*

In numbers, let there be AB, 5. CD, 45. CE = 36. ED = 9. Make 45. 9 :: 25 (ABq.) 5 (AFq.) therefore AF = $\sqrt{}$ 5. conſequently BFq = 45 − 25 = 20. wherefore BF = $\sqrt{}$ 20.

PROP. XXXII.

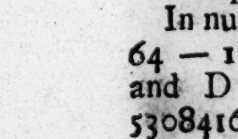


To find out two medial lines C, D, commenſurable only in power, comprehending a rational rectangle CD, ſo that the greater C be more in power than the leſſer D by the ſquare of a right line commenſurable in length to the greater.

a 30. 10. *a* Take A and B $\dot{\rho}$ ⌐⌐; ſo as $\sqrt{}$ Aq − Bq ⌐⌐
b 13. 6. A. *b* and make A. C :: C. B. *c* and A. B :: C.D.
c 12. 6. I ſay the thing is done.
d *conſtr.* For becauſe A and *d* B are $\dot{\rho}$ ⌐⌐. *e* therefore
e 22. 10. ſhall C (*f* $\sqrt{}$ AB) be μ. *g* and thence alſo C ⌐⌐ D.
f 17. 6. *b* therefore D is likewiſe μ. Furthermore, where-
g 10. 10. as A.B *d* :: C.D; and inverſely A.C :: B.D :: C.B;
h 24. 10. and Bq is $\dot{\rho}\nu$. therefore ſhall CD (*k* Bq) be $\dot{\rho}\nu$.
k 17. 6. Laſtly, becauſe $\sqrt{}$ Aq − Bq *d* ⌐⌐ A, *l* ſhall $\sqrt{}$ Cq
l 15. 10. − Dq be ⌐⌐ C. therefore, &c. But if $\sqrt{}$ Aq − Bq ⌐⌐ Aq, then ſhall $\sqrt{}$ Cq − Dq be ⌐⌐ C.

In numbers; let there be A 8, B $\sqrt{}$ 48 ($\sqrt{}$: 64 − 16) therefore C = $\sqrt{}$ AB = *v* $\sqrt{}$ 3072. and D = *v* $\sqrt{}$ 1728. wherefore CD = *v* $\sqrt{}$ 5308416 = $\sqrt{}$ 2304.

PROP. XXXIII.



To find out two medial lines D, E, commenſurable in power only, comprehending a medial rectangle DE, ſo far that the greater D ſhall be more in power than the leſs E, by the ſquare of a right line commenſurable to the greater in length.

a Take

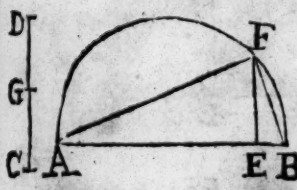
a Take A and C $\dot{\rho}$ ⊐, ſo that $\sqrt{Aq-Cq}$ ⊏ A. *b* take alſo B ⊐ A and C, and make A. D :: D. B *d* :: C. E. then D and E are the lines ſought for.

For becauſe A and C *e* are $\dot{\rho}$, *e* and B ⊐ A and C, *f* therefore ſhall B be $\dot{\rho}$, and D ($\sqrt{AB}$) *g* ſhall be μ. But becauſe A.D :: C.E, therefore inverſely A.C :: D. E. wherefore ſeeing A ⊐ C, therefore D ſhall be ⊐ E. therefore E is μ. Furthermore, *l* being D. B :: C. E. and BC is $\mu\nu$. alſo DE, equal to it, is $\mu\nu$. Laſtly, becauſe A. C :: D. E. *e* ſeeing $\sqrt{Aq-Cq}$ ⊏ A. therefore $\sqrt{Dq-Eq}$ ⊏ D. therefore, &c. But if $\sqrt{Aq-Cq}$ ⊏ A. then $\sqrt{Dq-Eq}$ ⊏ Eq.

In numbers, let there be A 8, C $\sqrt{48}$. B $\sqrt{8}$. then D v $\sqrt{3072}$. and E v $\sqrt{588}$. wherefore D. E :: 2. $\sqrt{3}$. and DE = $\sqrt{1344}$.

a 30. 10. b *lem.* 21. 10. c 13. 6. d 12. 6. e *conſtr.* f *ſch.* 12. 10. g 22. 10. h 10. 10. k 24. 10. l 22. 10. m 16. 6. n 15. 5.

PROP. XXXIV.



To find out two right lines AF, BF, incommenſurable in power, whoſe ſquares added together make a rational figure, and the rectangle contained under them medial.

a Let there be found AB, CD, $\dot{\rho}$ ⊐; ſo that $\sqrt{ABq-CDq}$ ⊏ AB. divide CD equally in G. *c* make the rectangle AEB = GCq. Upon AB the diameter draw the ſemicircle AFB, erect the perpendicular EF, and draw AF, BF. theſe are the lines required.

For AE. BE *d* :: BA × AE. AB × BE. But BA × AE *e* = AFq; and AB × BE = FBq. *f* therefore AE. EB :: AFq. FBq. therefore being AE *g* ⊏ EB, *h* AFq ſhall be ⊏ FBq. Moreover ABq (*k* AFq + FBq) *l* is $\dot{\rho}\nu$. Laſtly, EFq *l* = EB *l* = CGq. *m* therefore EF = CG. therefore CD × AB = 2 EF × AB. But CD × AB *n* is $\mu\nu$. *o* therefore AB × EF, *p* or AF × FB is $\mu\nu$. *Which was to be dem.*

a 31. 10. b 10. 1. c 28. 6. d 12. 6. e *cor.* 8. 6. & 17. 6. f 7. 5. g 19. 10. h 10. 10. k 31. 3. & 47. 1. l *conſtr.* m 1. *ax.* 1. n 22. 10. o 24. 10. p *ſch.* 22. 6.

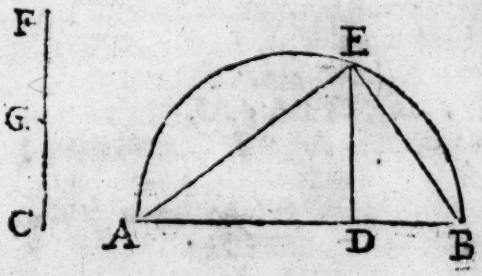
The

The Explication of the same by numbers.

Let AB be 6. CD $\sqrt{12}$. then CG $= \sqrt{\frac{12}{4}} = \sqrt{3}$. but AE $= 3 + \sqrt{6}$. and EB $= 3 - \sqrt{6}$. whence AF shall be $\sqrt{: 18 + 216}$. and FB $\sqrt{18 - 216}$. Also AFq + FBq is 36, and AF × FB $= \sqrt{108}$.

But AE is found in this manner. Because BA (6.) AF :: AF.AE. therefore 6 AE = AFq = AEq + 3 (EFq.) therefore 6 AE − AEq = 3. Put 3 + e = AE. then 18 + 6e − 9 − 6e − ee, that is, 9 − ee = 3. or ee = 6. wherefore e = $\sqrt{6}$ and so AE $= 3 + \sqrt{6}$.

PROP. XXXV.



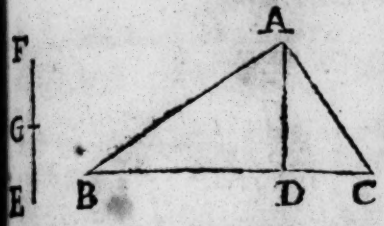
To find out two right lines AE, EB, incommensurable in power, whose squares added together make a medial figure, and the rectangle contained under them rational.

a 32. 10. *a* Take AB and CF μ ⊐, so that AB × CF be $\rho\nu$, and $\sqrt{}$ ABq − CFq ⊏ AB. and let the rest be done as in the prec. Prop. AE, EB are the line required.

For, as it is shewn there, AEq ⊏ EBq. also b *constr.* ABq (AEq + EBq) is $\mu\nu$, and lastly, AB × CF c *sch.* 12. *b* is $\rho\nu$, *c* therefore also AB × DE, that is, AE × d *sch.* 22. 6. EB, is $\rho\nu$. therefore, *&c.*

PROP.

PROP. XXXVI.

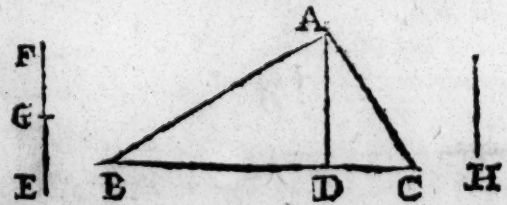


To find out two right lines BA, AC, incommenſurable in power, whoſe ſquares added together make a medial figure, and the rectangle alſo contained under them medial, and incommenſurable to the figure compoſed of the ſquares.

a Take BC and EF μ ⏋, ſo that BC × EF be $\mu\nu$. and $\sqrt{BCq - EFq}$ ⏍ BC. and ſo forward, as in the prec. BA, AB, ſhall be the lines ſought for. a 33. 10.

For (as above) BAq ⏍ ACq. alſo BAq+ACq is $\mu\nu$. and BA × AC is $\mu\nu$. Laſtly, BC *b* ⏍ EF, and *c* ſo BC ⏍ EG; likewiſe BC.EG *d*:: BCq. BC × EG (BC × AD, or BA × AC) *e* therefore BCq. (ABq+ACq) ⏍ BA × AC. therefore, *&c.* b *conſtr.* c 13. 10. d 1. 6. e 14. 10.

Schol.



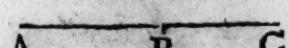
To find out two medial lines incommenſurable both in length and power.

a Take BC μ and let BA × AC be $\mu\nu$, and ⏍ BCq (BAq+ACq *b* make BA. H :: H. AC. then I ſay BC and H are μ ⏋. For BC is μ. *a* and BA × AC (*c* Hq) is $\mu\nu$. wherefore H is alſo μ. *d* Likewiſe BA × AC ⏍ BCq; therefore Hq ⏍ BCq. therefore, *&c.* a 36. 10. b 33. 6. c 17. 6. d 14. 10.

Here

Here begin the ſenaries of lines irrational by compoſition.

PROP. XXXVII.



If two rational lines AB, BC, commenſurable only in power, be added together, the whole line AC is irrational, and is called a binomial line, or of two names.

a *hyp.* For becauſe AB *a* ⊓ BC, thence *b* ſhall ACq
b *lem.* 26. 10. be ⊓ ABq. But AB *a* is a ῥ, *c* therefore AC
c 11. *def.* 10. is ῥ. *Which was to be dem.*

PROP. XXXVIII.

If two medial lines AB, BC, only in power commenſurable, be compounded, and contain a rational rectangle, the whole line AC is irrational, and called a firſt bimedial line.

a *hyp.* For being that AB *a* ⊓ BC, *b* ſhall ACq be
b *lem.* 26. 10. ⊓ AB × BC, ῥv. *c* therefore AC is ῥ. *Which*
c 11. *def.* 10. *was to be dem.*

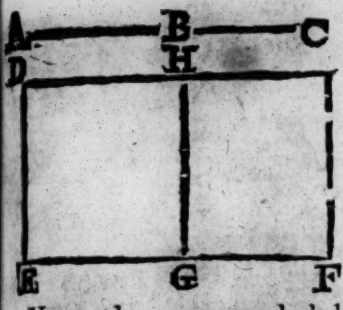
Lemma.

A rectangle AC, contained under a rational line AB and an irrational line BC, is irrational.

For if the rectangle AC be affirmed,
a *hyp.* ῥv, *a* then being AB is ῥ, *b* the breadth BC
b 21. 10. ſhall be alſo ῥ. *againſt the Hyp.*

PROP.

PROP. XXXIX.



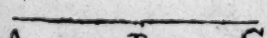
If two medial lines AB, BC, commensurable only in power, containing a medial rectangle, be compounded, the whole line AC shall be irrational, and is called a second bimedial line.

Upon the propounded line DE $\dot{\rho}$ *a* make the rectangle DF = ACq; *b* and DG = ABq + BCq.

Because ABq *c* $\sqsupset$ BCq, *d* therefore ABq + BCq, *i. e.* DG, $\sqsupset$ ABq: but ABq *e* is $\mu\nu$, *e* therefore DG is $\mu\nu$, but the rectangle ABC is taken $\mu\nu$; *e* and consequently 2 ABC (*f* HF) is $\mu\nu$. *g* therefore EG and GF are $\dot{\rho}$. Being also that DG *h* $\sqsupset$ HF; and DG. HF :: *k* EG. GF; *l* therefore EG $\sqsupset$ GF, *m* therefore the whole EF is $\dot{\rho}$. *n* wherefore the rectangle DF is $\dot{\rho}\nu$, *o* therefore $\sqrt{}$ DF, *i. e.* AC, is $\dot{\rho}$. *Which was to be dem.*

a *cor.* 10. 6.
b 47. 1. &
12. 6.
c *hyp.*
d 16. 10.
e 24. 10.
f 4. 2.
g 27. 10.
h *lem.* 26.
10.
k 1. 6.
l 10. 10.
m 37. 10.
n *lem.* 38.
10.

PROP. XL.



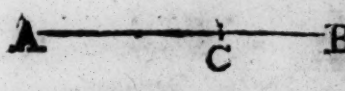
If two right lines AB, BC commensurable only in power, be added together, making that which is composed of their squares rational, and the rectangle contained under them medial, the whole right line AC is irrational, and is called a Major line.

For whereas ABq + CBq *a* is $\dot{\rho}\nu$. and *b* $\sqsupset$ 2 ABC *c* $\mu\nu$; and so ACq (*d* ABq + BCq + 2 ABC) *e* $\sqsupset$ ABq + BCq $\dot{\rho}\nu$, *f* therefore shall AC be $\dot{\rho}$. *Which was to be dem.*

a *hyp.*
b *sch.* 12.
10.
c *hyp.* and
24. 10.
d 4. 2.
e 17. 10.
f 11. *def.*
10.

PROP.

PROP. XLI.

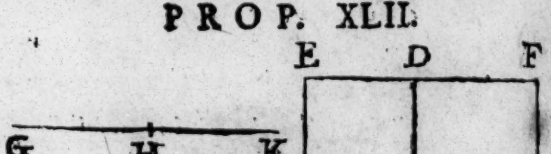


If two right lines AC, CB, incommensurable in power, be added together, having that which is made of their squares added together medial, and the rectangle contained under them rational, the whole right line AB shall be irrational, and is called A line containing in power a rational and a medial rectangle.

a *hyp.* and *sch.* 12. 10. b *sch.* 12. 10. c *hyp.* d 17. 10. e 11. *def.* 10.

For 2 rectangles ACB *a* $\dot{\rho}\nu$, *b* $\square$ ACq + CBq *c* $\mu\nu$. *d* therefore 2 ACB *d* $\square$ ABq. wherefore *e* AB is $\dot{\rho}$. *Which was to be dem.*

PROP. XLII.



If two right lines GH, HK, incommensurable in power be added together, having both that which is composed of their squares medial, and the rectangle contained under them medial, and incommensurable to that which is composed of their squares, the whole right line GK is irrational, and is called a line containing in power two medial figures.

a *hyp.* b 23. 10. c 4. 2. d 2. 6. e 20. 10. f 37. 10. g *lem.* 38. 10. h 21. *def.* 10.

Upon the propounded line FB $\dot{\rho}$ make the rectangles AF = GKq, and CF = GHq + HKq. Being GHq + HKq (CF) *a* is $\mu\nu$, the breadth CB *b* shall be $\dot{\rho}$. Also because 2 rectangles GHK (*c* AD) *a* is $\mu\nu$, therefore AC *b* shall be $\dot{\rho}$. Moreover because the rectangle AD *a* $\square$ CF, *d* and AD.CF :: AC. CB. *e* thence shall *e* AC be $\square$ CB. *f* wherefore AB is *g* $\dot{\rho}$. therefore the rectangle AF. *i. e.* GKq is $\dot{\rho}\nu$; *b* and consequently GK is $\dot{\rho}$. *Which was to be dem.*

PROP.

PROP. XLIII.

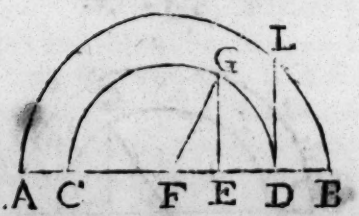
A line of two names, or binomial, AB, can at one point only D be divided into its names, AD, DB.

If it be poſſible, let the binomial line AB be divided at the point E, into other names AE, EB. It is manifeſt that the line AB is in both caſes divided unequally, ſince AD ⊐ DB, and AE ⊐ EB.

Becauſe the rectangles ADB, AEB *a* are $\mu\alpha$; *a* and each of ADq, DBq, AEq, EBq is $\rho\alpha$. *b* and ſo ADq + DBq *b* and AEq + EBq are alſo $\rho\alpha$. *b* therefore ADq + DBq − : AEq + EBq *c i. e.* 2 AEB − 2 ADB is $\rho\nu$. *d* therefore AEB − ADB is $\rho\nu$. therefore $\mu\nu$ exceeds $\mu\nu$ by $\rho\nu$. *e Which is abſurd.*

a 37. 10.
b *ſch.* 27. 10.
c *ſch.* 5. 2.
d *ſch.* 12. 10.
e 27. 10.

PROP. XLIV.

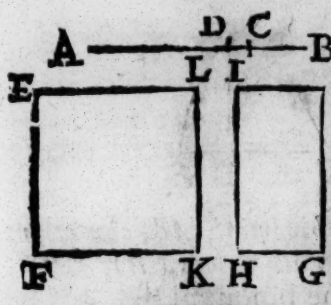


A firſt bimedial line AB, is in one point only D divided into its names AD, DB.

Conceive AB to be divided into other names AE, EB, whereupon every one ADq, DBq, EBq, will be *a* $\mu\alpha$. and the rectangles ADB, AEB, and the doubles of them $\rho\alpha$. *b* therefore 2 AEB − 2 ADB. *c i. e.* ADq + DBq − : AEq + EBq is $\rho\nu$. *Which is abſurd.*

a 38. 10.
b *ſch.* 27. 10.
c *ſch.* 5. 2.
d 27. 10.

PROP. XLV.

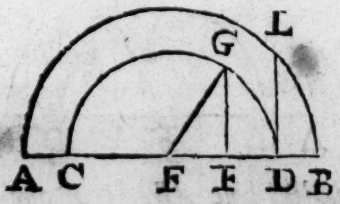


A ſecond bimedial line AB, is divided into its names AC, CB, only at one point C.

Suppoſe there were other names AD, DB. Upon the propounded line EF ῥ make the rectangles EG = ABq, and EH = ACq + CBq. as alſo EK = ADq + DBq.

a 39. 10. Becauſe ACq, BCq *a* are μα ▭; *b* ACq +
b 16. *and* CBq (EH) ſhall be μν. *c* therefore the breadth
24. 10. FH is ῥ. *a* moreover the rectangle ACB, *d* and ſo
c 23. 10. 2 ACB (*e* IG) is μν. *c* therefore HG is alſo ῥ.
d 24. 10. And ſince EH is *f* ▭ IG, *g* and EH.IG :: FH.
e 4. 2. HG. *h* therefore FH, HG ſhall be ▭. *k* there-
f *lem.* 26. fore FG is a binomial, whoſe names are FH,
10. HG. By the ſame reaſon FG is binomial, and
g 1. 6. the names of it FK, KG : *contrary to the* 43. *of*
h 10. 10. *this Book.*
k 37. 10.

PROP. XLVI.

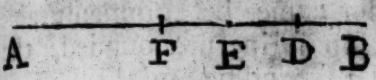


A Major line AB is at one point only D divided into its names, AD, DB.

Imagine other names AE, EB. whereupon the
a 40. 10. rectangles ADB, AEB, are *a* μα. *a* and as well
b *ſch.* 27. ADq + DBq, as AEq + EBq are ῥα. *b* therefore
10. ADq + DBq — : AEq + EBq, *c i. e.* 2 AEB —
c *ſch.* 5. 2. 2 ADB is ῥν. *Which is impoſſible.*
d 27. 10.

PROP.

PROP. XLVII.

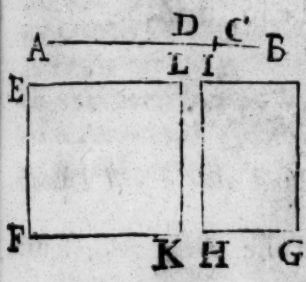


A line AB containing in power a rational and a medial figure is divided at one point only D into its names AD, DB.

Conceive other names AE, EB. then both AEq + EBq, and ADq + DBq are $\mu\alpha$. *a* and the rectangles AEB, ADB are $\rho\alpha$. *b* therefore 2 AEB − 2 ADB, *c i. e.* ADq + DBq − AEq + EBq is $\rho\nu$. *d Which is absurd.*

a 41. 10.
b *sch.* 17. 10.
c *sch.* 5. 2.
d 27. 10.

PROP. XLVIII.



A line AB containing in power two medial rectangles, is at one point only C divided into its names AC, CB.

If you will divide AB into other names AD, DB. draw upon the line propounded EF ρ the rectangles EG = ABq, and EH = ACq + CBq, and EK = ADq + DBq. then because ACq + CBq, namely, EH, *a* is $\mu\nu$, *b* the breadth FH shall be ρ. Also because 2 ACB, *c* that is, IG, is *a* $\mu\nu$, HG *b* shall be likewise ρ. Therefore, whereas EH *a* ⊓ IG. and FH. IG *d* :: FH. HG, thence FH *e* shall be ⊓ HG. *f* therefore FG is a binomial, and the names of it FH, HG. In like manner FK, KG shall be the names of it, *against the* 43. *of this Book.*

a 42. 10.
b 23. 10.
c 4. 2.
d 1. 6.
e 10. 10.
f 37. 10.

Second Definitions.

A Rational line being propounded, and the binomial divided into its names, the greatest of whose names is more in power than the less by the square of a right line commensurable to the greater in length; then

I. If the greater name be commensurable in length to the rational line propounded, the whole line is called a first binomial line.

II. But if the lesser name be commensurable in length to the rational line propounded, the whole line is called a second binomial.

III. If neither of the names be commensurable in length to the rational line propounded, it is called a third binomial.

Furthermore if the greater name be more in power than the less by the square of a right line incommensurable to the greater in length, then

IV. If the greater name be commensurable to the propounded rational line in length, it is called a fourth binomial.

V. If the lesser name be so, a fifth.

VI. If neither, a sixth.

PROP. XLIX.

A....4C.....5B
D ——.——
E ——— ——— G
H —— F

To find out a first binomial line, EG.

a Take AB, AC, square numbers, whose excess CB is not Q. let D be propounded ρ̊. *b* Take EF ⊓ D, and *c* make AB. CB :: EFq, FGq. then EG shall be a 1. bin.

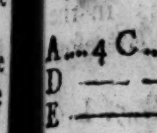
For EF *d* ⊓ D. *e* therefore EF is ρ̊. *f* also EFq ⊓ FGq. *g* therefore FG is also ρ̊. likewise *d* be EFq. FGq :: AB.CB :: Q. not Q. *h* therefore EF ⊓ FG. Lastly, because by conversion of proportion, EFq, EFq − FGq :: AB.AC :: Q.Q. thence EF *k* shall be ⊓ √ EFq — FGq. *l* therefore EG is a first binomial. *Which was to be done.*

In numbers thus; let there be D 8. EF 6. AB 9. CB 5. wherefore because 9. 5 :: 36. 20, therefore FG is √ 20. and consequently EG is 6 + √ 20.

a *sch.* 29. 10.
b 2. *lem.* 10. 10.
c 3. *lem.* 10. 10.
d *constr.*
e 6. *def.* 10.
f 6. 10.
g *sch.* 12. 10.
h 9. 10.
k 9. 10.
l 1. *def.* 48. 10.

PROP.

PROP. L.



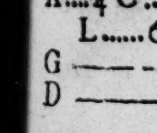
To find out a ſecond binomial line, EG.

Take AB and AC ſquare numbers, the exceſs of which is CB not Q. Let D be the line propounded ṕ. take FG ⊐ D, and make CB. AB :: FGq. EFq. then EG will be the line deſired. *Prove it as the prec.*

For FG ⊐ D. wherefore FG is ṕ. Alſo EFq ⊐ FGq. therefore EF is ṕ. Likewiſe becauſe FGq. EFq :: CB, AB :: not Q. Q. thence FG is ⊐ EF. Laſtly, ſeeing CB. AB :: FGq. EFq. and inverſely AB. CB :: EFq. FGq. therefore as in the foregoing Prop. EF ⊐ √ EFq − FGq. *a* whereby EG is a 2. binomial. *Which was to be done.* a 2. *def.* 48. 10.

In numbers; let there be D 8, FG 10, AB 9, CB 5, then EF is √ 180. wherefore EG is 10 + √ 180.

PROP. LI.



To find out a third binomial line, DF.

a Take A B, A C, ſquare numbers, the exceſs of which B C is not Q. and let L be a number not Q next greater than CB, *viz.* by a unit or two. Let G be the line propounded ṕ. *b* Make L. AB :: Gq, DEq. *b* and AB. CB :: DEq. EFq. then DF ſhall be a 3. bin. a *ſch.* 19. 10. b 3. *lem.* 10. 10.

For becauſe DEq *c* ⊐ Gq, *d* DE is ṕ. alſo Gq. DEq :: L. AB :: not Q. Q. *e* therefore G ⊐ DE. Likewiſe being that DEq *e* ⊐ EFq, *d* alſo EF is ṕ. Moreover becauſe DEq. EFq :: AB. CB :: Q. not Q. *f* is DE ⊐ EF. and being that by conſtr. and c *conſtr.* 6. 10. d *ſch.* 12. 10. e 6. 10.

f 9. 10. of equality Gq. EFq :: L. CB :: not Q. Q. (for
g *ſch.* 27. 8. L and CB are not like plane numbers) *h* there-
h 9. 10. fore ſhall G be alſo ▭ EF. Laſtly, as in the
prec. Prop. √ DEq — EFq ▭ DE. *k* therefore
k 3. *def.* 48. DF is a 3 bin. *Which was to be done.*

10. In numbers; let there be AB, 9. CB, 5. L, 6. G, 8. then ſhall be DE √ 96, and EF √ $\frac{480}{9}$. wherefore DF = √ 96 + √ $\frac{480}{9}$.

PROP. LII.

A ... 3 C ..... 6 B
G ——— ——
D ——— ——— F
H ——— E

To find out a fourth binomial line, DF.

a *ſch.* 29. 10. *a* Take any ſquare number AB, and divide it into AC, CB not ſquares. Let G be the
b 2. *lem.* 10. line propounded ῥ. *b* take DE ▭ G. *c* and make
c 3. *lem.* 10. 10. AB. CB :: DEq. EFq. then DF ſhall be a 4 bin.
d 9. 10. For, as in the 49 of this Book, DF may be ſhewn to be a binom. and alſo becauſe by conſtr. and converſion of proportion DEq. DEq — EFq
e 4. *def.* 48. 10. :: AB. AC :: Q not Q. *d* ſhall DE be ▭ √ DEq — EFq. *e* therefore DF is a 4 bin.

In numbers, let G be 8, DE, 6. then EF ſhall be √ 24. therefore DF is 6 + √ 24.

PROP. LIII.

A ... 3 C ..... 6 E
G ——— —
D ——— ——— F
H ——— E

To find out a fifth binomial line, DF.

Take any ſquare number AB, whoſe ſegments AC, CB are not Q. Let G be the line propounded ῥ. take EF ▭ G. and make CB. AB :: EFq. DEq. then ſhall DF be a 5 bin.

For

For DF ſhall be a bin. as in the 50. of this Book, and becauſe by conſtruction, and inverſion DEq. EFq :: AB. CB. and ſo by converſion of proportion, DEq. DEq − EFq :: AB. AC :: Q. not Q. *a* therefore ſhall DE be ⊏ √ DEq − EFq. *b* therefore DF is the 5 bin. *Which was to be done.*

a 9. 10.
b 5. *def.* 48. 10.

In numbers, let there be G,7. EF,6. then DE ſhall be √ 54. wherefore DF is 6 + √ 54.

PROP. LIV.

To find out a ſixth binomial line.

Take AC,CB, prime numbers, ſo that AC + CB (AB) be not Q. take alſo any number ſquare L. Let G be the line propounded ſ. *a* and make L. AB :: Gq. DEq. and AB. CB :: DEq. EFq. then DF ſhall be a 6 binomial.

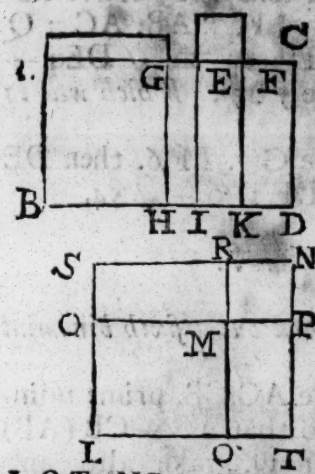
a 3. *lem.* 10. 10.

For DF may be demonſtrated bin. as in the 51. of this Book, and alſo by reaſon that DE and EF ⊏ G. laſtly, likewiſe becauſe by conſtr. and converſion of proportion DEq. DEq − EFq :: AB. AC :: not Q. Q. (For AB is prime to AC, *b* and ſo unlike to it) *c* therefore DE ⊏ √ DEq − EFq. *d* therefore DF is a 6 bin. *Which was required.*

b *ſch.* 27. 8.
c 9. 10.
d 6. *def.* 48. 10.

In numbers, let there be G 6. DE √ 48. then EF ſhall be √ 28. wherefore DF is √ 48 + √ 28.

Lemma.



Let AD be a rectangle, and the sides thereof AC divided unequally in E; also let the leſſer portion EC be equally divided in F, upon the line AE a *make the rectangle AGE = EFq. and from the points G, E, F* b *draw GH, EI, FK, parallel to AB.* c *Let the ſquare LM be made equal to the rectangle AH, and upon OMP produced the ſquare MN = GI, and let the right lines LOS, LQT, NRS, NPT be produced.*

a 28. 6.
b 31. 1.
c 14. 2.

I ſay 1. MS, MT, *are rectangles.* For by reaſon of the right angles of the ſquares QMO, RMP, *a* ſhall QMR be a right line. *b* therefore RMO, QEP, are right angles. wherefore the parallelograms MS, MT are rectangles.

a *ſch.*15.1.
b 13. 2.

2. *Hence it is plain that LS c = LT, and conſequently that LN is a ſquare.*

c 2. *ax.*21.

3. *The rectangles SM, MT, EK, FD are equal.* For becauſe the rectangle AGE *d* = EFq. *e* thence ſhall AE. EF :: EF. GE. *f* and ſo AH. EK :: EK. GI. that is by conſtr. LM. EK :: EK. MN. *g* but LM. SM :: SM. MN. therefore EK *h* = SM *k* = FD *l* = MT.

d *hyp.*
e 17. 6.
f 1. 6.
g *ſch.*22.6.
h 9. 5.
k 36. 1.

3. *Hence LN m = AD.*

l 43. 1.
m 2. *ax.*1.

5. *Being that EC is equally divided in F, it is plain that EF, FC, EC are* ▭.

n 16. 10.
o 18. *and* 16. 10.
p 10. 10.

6. If AE ▭ EC, and AE ▭ √ AEq - ECq, then ſhall AG, GE, AE, be ▭ alſo, becauſe AG. GE :: AH. GI. *p* therefore ſhall AH, GI, *i. e.* LM, MN, be ▭. Likewiſe thereupon,

7. OM

7. OM ⊓ MP. For by the Hyp. AE ⊓ EC. *q* therefore BC ⊓ GE. *q* wherefore EF ⊓ GE. but EF. GE :: EK. GI. *r* therefore EK ⊓ GI, that is, SM ⊓ MN. but SM. MN :: OM. MP. *r* therefore OM ⊓ MP. q 14. 10. r 10. 10.

8. If AE be supposed ⊓ √ AEq − ECq, it is apparent that AG, GE, AE, are ⊓. whence LM ⊓ MN. for AG. GE :: AH. GI :: LM. MN. s 19. *and* 17. 10.

These being well considered, we shall easily dispatch the six following Propositions.

PROP. LV.

If a space AD be contained under a rational line AB, and a first binomial line AC (AE + EC) the right line OP which containeth that space in power is irrational, and called a binomial line.

All that being supposed which is described and demonstrated in the next foregoing *Lemma*, it is manifest that the right line OP containeth in power the space AD. *a* Likewise AG, GE, AE are ⊓. therefore seeing AE is ῥ ⊓ AB. *c* shall also AG and GE be ῥ ⊓ AB. *d* therefore the rectangles AH, GI, that is the squares LM. MN are ῥα. therefore OM, MP are ῥ *e* ⊓. *f* and consequently OP is a binomial. *Which was to be dem.*

a *hyp. and lem.* 54. 10.
b *hyp.*
c *sch.* 12. 10.
d 20. 10.
e *lem.* 54.

In numbers, let there be AB 5. AC 4 + √ 12. wherefore the rectangle AD = 20 + √ 300 = to the square LN. therefore OP is √ 15 + √ 5. namely a 6 binomial. f 37. 10.

PROP. LVI.

If a space AD be comprehended under a rational line AB, and a second binomial AC (AE + EC) the right line OP, which contains that space AD in power, is irrational, and called a first medial line.

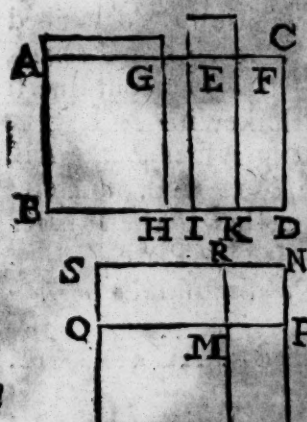
The

The foresaid *Lemma* of the 54 of this Book being again supposed, then shall OP be = √ A
a *hyp. and lem.* 54. 10. *a* also AE, AG, GE are ⊐ therefore since AE
b *hyp.* is ῥ ⊏ AB, likewise AG, GE *c* shall be ῥ ⊐
c *sch.* 12. 10. AB. therefore the rectangle AH, GI, *i. e.* OM
MPq. *d* are *ua. e* Moreover OM ⊐ MP. L
e *lem.* 54. 10. ly, EF ⊏ EC, and EC *f* ⊏ AB, *g* wherefo
f *hyp.* 12. 10. EK. *i. e.* SM, or OMP is ῥ*r*. *h* Consequently O
is a first bmedial. *Which was to be dem.*

In numbers, let there be AB, 5; and AC √
g 20. 10. 48: +6. then the rectangle AD = √: 1200 + 3
h 38. 10. = OPq. therefore OP is *v* √ 675 + *v* √ 75. *viz.* a first bimedial.

See Scheme 57.

PROP. LVII.



If a space AD be contained under a rational line AB and a third binomial line AC (AE+EC) the right line OP which contains in power the space AD, is irrational, and called a second bimedial line.

As above, OPq = AD. also the rectangles AH, GI, that is OMP, MPq are *ua.* *a* Likewise EK or OMP is *μγ.* *b* therefore OP is a second bimedial.

a *hyp. and* 22. 10.
b 39. 10.

In numbers; let there be AB 5. AC √ 32 + √ 34. wherefore AD is √ 800 + √ 600 = OPq. and so OP is *v* √ 450 + *v* √ 50. that is a 2. bimed.

PROP.

PROP. LVIII.

If a ſpace AD be comprehended under a rational line AB and a fourth binomial AC (AE + EC) the right line OP containing the ſpace AD in power, is that irrational line which is called a Major line.

For again, OMq *a* □ MPq; and the rectangle AI, *i. e.* OMq + MPq *b* is ῥν. *c* alſo EK or OMP is μν. *d* therefore OP (√ AD) is a Major line. *Which was to be dem.*

a *lem.* 54. 10.
b *hyp. and* 20. 10.
c *hyp. and* 22. 10.
d 40. 10.

In numbers; let there be AB 5. and AC 4 + √8. then the rectangle AD is 20 + √200. wherefore OP is √: 20 + √200.

PROP. LIX.

If a ſpace AD be contained under a rational line AB, and a fifth binomial AC, the right line OP which contains the ſpace AD in power, is that irrational line, which is a line containing a rational and a medial rectangle in power.

Again OMq □ MPq. and the rectangle AI or OMq + MPq is μν. *a* Likewiſe the rectangle EK or OMP is ῥν, *b* therefore OP (√ AD) contains in power ῥν and μν. *Which was to be dem.*

a *As in the prec.*
b 41. 10.

In numbers, let there be AB 5. and AC 2 + √8. then the rectangle AD = 10 + √200 = OPq, wherefore OP is √: 10 + 200.

PROP.

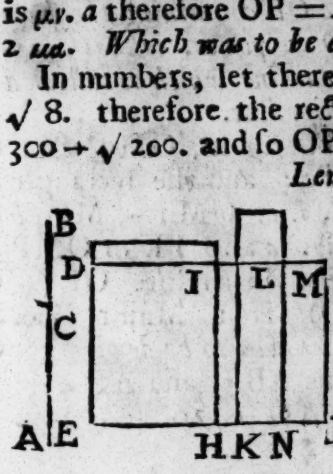
PROP. LX.

If a ſpace AD be contained under a rational line AB and a ſixth binomial AC (AE + EC) the line OP containing the ſpace AD in power is irrational, which contains in power two medial rectangles.

As often before, OMq □ MPq, and OMq + MPq is μν. and alſo the rectangle (EK) OMP
a 42. 10. is μν. *a* therefore OP = √ AD contains in power 2 μα. *Which was to be dem.*

In numbers, let there be AB 5. AC √ 12 + √ 8. therefore the rectangle AD or OPq is √ 300 + √ 200. and ſo OP is √: √ 300 + √ 200.

Lemma.



Let a right line AB be unequally divided in C, and let AC be the greater portion, and upon ſome line DE apply the rectangles DF = ABq, and DH = ACq, and IK = CBq. and let LG be divided equally in M, and alſo MN drawn parallel to GF.

a 4.2. *and* 3. *ax.* 1. I ſay 1. The rectangle ACB is = LN or MF. *a* For 2 ACB = LF.

b 7. 2. 2. DL ⊏ LG. for DK (ACq + CBq) *b* ⊏ LF
c 1. 6. (2 ACB) therefore being DK, LF are of equal
d 16. 10. altitude, *c* DL ſhall be ⊏ LG.

3. If AC □ CB, *d* then ſhall the rectangle DK be □ ACq and CBq.

e *lem.* 26. 10. 4. *Alſo* DL □ LG. for ACq + CBq *e* □ 2 ACB, *i. e.* DK □ LF. but DK. LF *e* :: DL.
f 10. 10. LG. *f* therefore DL □ LG.

g 1. 6. 5. *Moreover* DL □ √ DLq − LGq. For
h 17. 6. ACq. ACB *g* :: ACB. CBq. that is, DH. LN ::
k *hyp.* LN. IK. *e* wherefore DI. LM :: LM. IL. *h*
l 10. 10. therefore DI × IL = LMq. therefore ſeeing ACq
m 18. 10. *k* □ CBq, that is, DH □ IK. and *l* ſo DI □ IL. *m* ſhall DL be □ √ DLq − LGq. *Which was to be dem.*

6. *But*

6. *But if ACq be put* ▭ *CB, then ſhall DL be* ▭ √ *DLq* — *LGq.* n 19. 10.

This Lemma is preparatory to the following Propoſitions.

PROP. LXI.

The ſquare of a binomial line (AC + CB) applied unto a rational line DE, makes the breadth DG a firſt binomial line.

Thoſe things being ſuppoſed, which are deſcribed and demonſtrated in the next preceding *Lemma*; becauſe AC, CB *a* are ῥ ▭, *b* the rectangle DK ſhall be ▭ ACq. *c* and ſo DK is ῥν. *d* therefore DL ▭ DE ῥ. but the rectangle ACB, and ſo 2 ACB (LF) *e* is μν. *f* therefore the latitude LG is ῥ ▭ DE. *g* therefore alſo DL ▭ LG alſo DL ▭ √ DLq - LGq. from whence *k* it follows that DG is a firſt binomial. *Which was to be dem.*

a *hyp.* b *lem.* 60. 10. c *ſch.* 12. d 21. 10. e 22. *and* 24. 10. f 23. 10. g 13. 10. h *lem.* 60. 10. k 1. *def.* 10. 10.

PROP. LXII.

The ſquare of a firſt bimedial line (AC + CB) being applied to a rational line DE, makes the breadth DG a ſecond binomial line.

The aforeſaid *Lemma* being again ſuppoſed; the rectangle DK ▭ ACq. *a* therefore DK is μν. *b* therefore the breadth DL is ῥ ▭ DE. But becauſe the rectangle ACB, and ſo LF (2 ACB) *c* is ῥν, *d* ſhall LG be ῥ ▭ DE. *e* therefore DL, LG are ▭ *f* alſo DL ▭ √ DLq — LGq. *g* from whence it is clear that DG is a ſecond binomial. *Which was to be dem.*

a 24. 10. b 23. 10. c *hyp. and ſch* 22. 10. d 2. 10. e 13. 10. f *lem.* 10. 10. g 2 *def.* 48. 10.

PROP.

PROP. LXIII.

The ſquare of a ſecond bimedial line (AC+CB) applied to a ratienal line DE makes the breadth DG a third binomial line.

a *hyp. and* 24. 10. b 23. 10. c *lem.* 60. 10. d 3. *def.* 48. 10.

As in the prec. DL is $\dot{\rho}$ ⊥ DE. Furthermore becauſe the rectangle ACB, and ſo LF (2 ACB) *a* is $\mu\nu$. *b* therefore ſhall LG be $\dot{\rho}$ ⊥ DE. *c* Moreover DL ⊥ LG. and alſo DL ⊥ √ DLq — LGq. *d* therefore DG is a third binomial. *Which was to be dem.*

PROP. LXIV.

The ſquare of a Major line (AC+CB) applied to a rational line DE, makes the breadth DG a fourth binomial line.

a *hyp. and ſch.* 12. 10. b 21. 10. c *hyp. and* 24. 10. d 23. 10. e 13. 10. f *lem.* 60. 10. g 4. *def.* 48. 10.

Again ACq+CBq. *i. e.* DK *a* is $\dot{\rho}\nu$ *b* therefore DL is $\dot{\rho}$ ⊥ DE, alſo ACB, and ſo LF (2 ACB) *c* is $\mu\nu$. *d* therefore LG is $\dot{\rho}$ ⊥ DE, *e* and conſequently DL ⊥ LG. Laſtly, becauſe AC ⊥ BC. *f* ſhall DL be ⊥ DLq - LGq. *g* whence DG is a fourth binomial. *Which was to be dem.*

PROP. LXV.

The ſquare of a line containing in power a rational and a medial rectangle (AC + CB) applied to a rational line DE makes the breadth DG a fifth binomial.

a 23. 10. b 21. 10. c 13. 10. d *lem.* 60. 10. e 5. *def.* 48. 10.

Again, DK is $\mu\nu$. *a* therefore DL is $\dot{\rho}$ ⊥ DE. alſo LF is $\dot{\rho}\nu$. *b* therefore LG is $\dot{\rho}$ ⊥ DE. *c* therefore DL ⊥ LG. *d* likewiſe DL ⊥ √ DLq—LGq. *e* and ſo by conſequence DG is a fifth binomial. *Which was to be dem.*

PROP. LXVI.

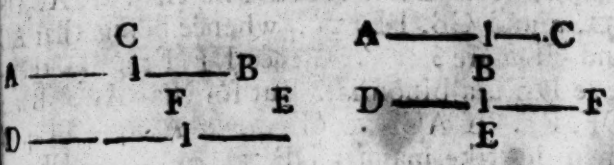
The ſquare of a line containing in power two medial rectangles (AC+CB) applied to a rational line DE, makes the breadth DG a ſixth binomial line.

As

As before, DL and LG are *i* ⊐ DE. But for that ACq + CBq (DK) *a* □ ACB, *b* and ſo DK □ LF (2 ACB) and alſo DK. LF *c* :: DL. LG. *d* therefore ſhall DL be □ LG. *e* Laſtly, DL □ √ DLq — LGq. *f* by which it appears that DG is a ſixth binomial.

a *hyp.* b 14. 10. c 1. 6. d 10. 10. e *lem.* 60. 10. f 6. *def.* 48. 10.

Lemma.



Let AB, DE be □. *and make AB. DE :: AC. DF.*

I ſay 1. AC □ DF. as appears by 10. 10. alſo CB □ FE. *a* becauſe AB. DE :: CB. FE. a 19. 5.

2. AC. CB :: DF. FE. For AC. DF :: AB. DE :: CB. FE. therefore inverſely AC. CB :: DF. FE.

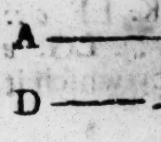
3. *The rectangle* ACB □ DFE. For ACq. ACB *b* :: AC. CB *c* :: DF. EF :: DFq. DFE. wherefore by inverſion ACq. DFq :: ACB. DFE. therefore being ACq □ DFq. *d* ſhall ACB be □ DFE. b 1. 6. c *before* d 10. 10.

4. ACq + CBq □ DFq + EFq. For becauſe ACq. CBq *e* :: DFq. FEq. therefore by addition ACq + CBq. CBq :: DFq + FEq. FEq. therefore being CBq □ FEq, *f* ſhall alſo ACq + CBq be □ DFq + FEq. e 22. 6. f 10. 10.

5. *Hence*, If AC ⊐ or ⊐ CB. *g* then likewiſe ſhall DE be ⊐ or ⊐ EF. g 10. 10.

PROP.

PROP. LXVII.



A line DE, commenſurable in length to a binomial line (AC+CB) is it ſelf a binomial line, and of the ſame order.

Make AB. DE :: AC. DF. *a* then are AC, DF and CB, FE ▭. whence being that AC and CB *b* are $\dot{\rho}$ ⊐, *c* thence DF, FE $\dot{\rho}$ ⊐ therefore DE is a binomial. But for that AC. CB *a* :: DF. FE. if AC ▭ or ▭ $\sqrt{}$ ACq − BCq. *d* then in like manner DF ▭ or ▭ $\sqrt{}$ DFq − FEq. alſo if AC ▭ or ▭ $\dot{\rho}$ propounded, *e* then ſhall DF be ▭ or ▭ $\dot{\rho}$ propounded. But if CB ▭ or ▭ $\dot{\rho}$. likewiſe FE ▭ or ▭ $\dot{\rho}$. If both AC, CB, ▭ $\dot{\rho}$. *g* that is, whatſoever binomial AB is, DE ſhall be of the ſame order. *Which was to be dem.*

a *lem.* 66. 10.
b *hyp.*
c *lem.* 66. 10. *and ſch.* 12. 10.
d 15. 10.
e 12. 10. *and* 14. 10.
f *by def.* 48. 10.
g 14. 10.

PROP. LXVIII.

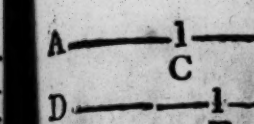
A line DE commenſurable in length to a bimedial line (AC+CB) is alſo a bimedial line, and of the ſame order.

Make AB. DE :: AC. DF. *b* therefore AC ▭ DF and CB ▭ FE. therefore ſeeing AC and CB *c* are μ, *d* alſo DF, and FE ſhall be μ, and for that AC *e* ⊐ CB. *e* therefore FD ⊐ FE. *f* therefore DE is 2 μ. Wherefore if the rectangle ACB be $\dot{\rho}\nu$. becauſe DFE *b* ▭ ACB, *g* likewiſe DFE is $\dot{\rho}\nu$. and if that be $\mu\nu$. *h* this ſhall be $\mu\nu$ too. *k* That is, whether AB be 1 bimed. or 2 bimed. DF ſhall be of the ſame order. *Which was to be dem.*

a 12. 6.
b *lem.* 66. 10.
c *hyp.*
d 24. 10.
e 10. 10.
f 38. 10.
g *ſch.* 12. 10.
k 38, *or* 39 10.

PROP.

PROP. LXIX.



A line DE commenſurable to a Major line (AC + CB) is it ſelf a Major line.

Make AB. DE :: AC. DF. Becauſe AC *a* ⊐ CB, *b* thence DF ⊐ FE. Alſo ACq + CBq *a* is ῥν. and ſo being DFq + FEq *b* ▭ ACq + CBq, *c* alſo DFq + FEq is ῥν. laſtly, the rectangle ACB *a* is μν. *d* therefore the rectangle DFE is μν. (becauſe DFE is *b* ▭ ACB) *e* wherefore DE is a Major line.

a *hyp.*
b *lem.* 66. 10.
c *ſch.* 12. 10.
d 24. 10.
e 40. 10.

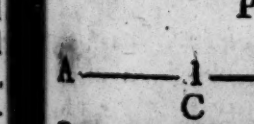
PROP. LXX.

A line DE commenſurable to a line containing in power a rational and a medial rectangle (AC + CB) is a line containing in power a rational and a medial rectangle.

Again make AB. DE :: AC. DF. Becauſe AC *a* ⊐ CB, *b* alſo DF ⊐ FE. likewiſe becauſe ACq + CBq *a* is μν, *c* therefore DFq + FEq ſhall be μν. laſtly, becauſe the rectangle ACB *a* is ῥν, *d* alſo DFE is ῥν. Therefore DE contains in power ῥν and μν. *Which was to be dem.*

a *hyp.*
b *lem.* 66. 10.
c 24. 10.
d *ſch.* 12. 10.
e 41. 10.

PROP. LXXI.



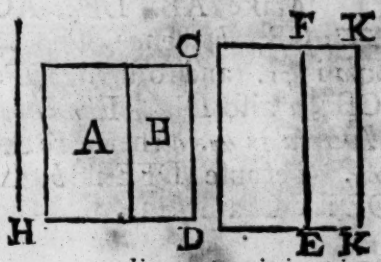
A line DE commenſurable to a line containing two medial rectangles in power (AC + CB) is alſo a line containing in power two medial rectangles.

Divide DE, as in the prec. Becauſe ACq *a* ▭ CBq, *b* thence ſhall DFq be ▭ FEq. alſo for that ACq + CBq *a* is μν, *c* ſhall DFq + FEq be alſo μν. And in like manner becauſe ACB *a* is μν. *d* alſo DFE is μν. Laſtly, becauſe ACq + CBq ▭ ACB,

a *hyp.*
b *lem.* 66. 10.
c 24. 10.
d 24. 10.

e 14. 10. *e* ſhall DFq . FEq be ⊏ DFE. *f* from whence
f 42. 10. it follows that DE contains in power 2 μα. *Which was to be dem.*

PROP. LXXII.

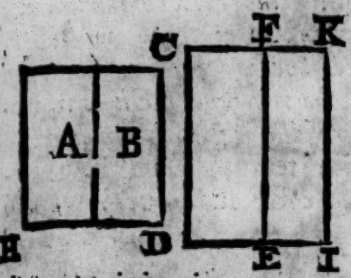


If a rational rectangle A, and a medial B, be compoſed together, theſe four irrational lines will be made; either a binomial, or a firſt bimedial, or a major, or a line containing in power a rational and a medial rectangle.

Namely, if Hq = A ∣ B. then H ſhall be one of the four lines which the Theorem mentions. For
a *cor.* 16. 6. upon CD the propounded $\dot{\rho}$, *a* make the rectan-
b 2. *ax.* 1. gle CE = A, and FI = B. *b* and ſo CI = Hq.
c 21. 10. Whereas then is A $\dot{\rho}\nu$, likewiſe CE is $\dot{\rho}\nu$. *c* there-
d 23. 10. fore the latitude CF is $\dot{\rho}$ ⊏ CD. and becauſe B
e 13. 10. is $\mu\nu$, alſo FI ſhall be $\mu\nu$. *d* therefore FK is $\dot{\rho}$ ⊏
f 37. 10. CD. *e* therefore CF, FK are $\dot{\rho}$ ⊏. and ſo the
g 1. 6. whole CK *f* is binom. wherefore if A ⊏ B, *i. e.*
h 1. *def.* 48. CE ⊏ FI, *g* then CF ⊏ FK. therefore if CF ⊏
10. √ CFq − FKq, *h* likewiſe CK ſhall be a 1 bin.
k 55. 10. and conſequently H = √ CI *k* is a bin. If CF
l 4. *def.* 48. be ſuppoſed ⊏ √ CFq − FKq, *l* then ſhall CK
10. be a 4 bin. wherefore H (√ CI) *m* is a major
m 58. 10. line. But if A ⊐ B, *g* then ſhall CF be ⊐ FK,
n 2. *def.* 48. conſequently if FK ⊏ √ FKq − CFq, *n* then
10. ſhall CK be a 2 bin. *o* wherefore H is a firſt 2 μ.
o 56. 10. laſtly, if FK ⊏ √ FKq − CFq ; *p* then CK
p 5. *def.* 48. ſhall be a fifth binom. *q* whence H ſhall contain
10. in power $\dot{\rho}\nu$ and $\mu\nu$. *Which was to be dem.*
q 59. 10.

PROP.

PROP. LXXIII.



If two medial rectangles A, B, incommensurable to one another be composed together, the two remaining irrational lines are made, either a second bimedial, or a line containing in power two medial rectangles.

As H containing in power A + B is one of the said irrational lines. For upon CD propounded $\dot{\rho}$ draw the rectangle CE = A, and FI = B. whence Hq = CI. Therefore because CE and FI *a* are $\mu\alpha$. *b* the latitudes CF, FK, shall be $\dot{\rho}$ ⊥ CD. also because CE *a* ⊥ FI, and CE. FI *c* :: CF. FK, *d* therefore CF ⊥ FK. *e* therefore CK is a 3. bin. namely, if CF ⊥ $\sqrt{\ }$ CFq − FKq. whence H = $\sqrt{\ }$ CI *f* shall be a second 2 μ. But if CF ⊥ $\sqrt{\ }$ CFq − FKq, *g* then CK shall be a 6 binom. *h* and consequently H contains in power 2 $\mu\alpha$. *Which was to be dem.*

a *hyp.*
b 23. 10.
c 1. 6.
d 10. 10.
e 3. *def.* 48.
f 57. 10.
g 6. *def.* 48.
h 60. 10.

Here begin the Senaries of lines irrational by Subtraction.

PROP. LXXIV.

D E F

If from a rational line DF a rational line DE, commensurable in power only to the whole DF, be taken away, the residue EF is irrational, and is called an Apotome or residual line.

For EFq *a* ⊥ DEq; *b* but DEq is $\dot{\rho}v$; *c* therefore EF is $\dot{\rho}$. *Which was to be dem.*

a *lem.* 26. 10.
b *hyp.*
c 10. & 11. *def.* 10.

In numbers, let there be DF, 2. DE, $\sqrt{3}$. then EF shall be $2 - \sqrt{3}$.

PROP. LXXV.

If from a medial line DF, a medial line DE commensurable only in power to the whole DF, and comprehending with the whole DF a rational rectangle be taken away, the remainder EF is irrational, and is called a first residual line of a medial.

a *sch.* 16. 10. b *hyp.* c 20. *and* 11. *def.* 10.

For EFq *a* ▭ to the rectangle FDE. therefore seeing FDE *b* is $\dot{p}v$, *c* BF shall be $\dot{p}$. *Which was to be dem.*

In numbers, let DF be $v\sqrt{54}$, and DE $v\sqrt{24}$. therefore EF is $v\sqrt{54} - v\sqrt{24}$.

PROP. LXXVI.

If from a medial line DF, a medial line DE be taken away being incommensurable only in power to the whole DF, and comprehending together with the whole line DF a medial rectangle, the remainder EF is irrational, and is called a second residual of a medial line.

a *hyp.* b 16. 10. c 24. 10. d *cor.* 7. 1. e 27. 10.

Because DFq and DEq *a* are $\mu\alpha$ ▭, *b* therefore shall DFq + DEq be ▭ DEq. *c* wherefore DFq + DEq is μv. also the rectangle FDE, *c* and so 2 FDE, *a* is μv. therefore EFq (*d* DFq + DEq − 2 FDE) *e* is $\dot{p}v$. wherefore EF is $\dot{p}$. *Which was to be dem.*

In numbers, let DF be $v\sqrt{18}$. and DE $v\sqrt{8}$. then EF $v\sqrt{18} - v\sqrt{8}$.

PROP. LXXVII.

If from a right line AC be taken away a right line AB being incommensurable in power to the whole BC, and making with the whole AC that which is composed of their squares rational, and the rectangle contained under them medial, the remainder BC is irrational, and is called a Minor line.

a *hyp.* b *sch.* 12. 10. c 7. 2.

For ACq + ABq *a* is $\dot{p}v$, but the rectangle ACB *a* is μv. *b* therefore 2 CAB ▭ ACq + ABq (2 c CAB

CAB+BCq.) *d* therefore ACq+ABq ⊏ BCq. *e* therefore BC is ṗ. *Which was to be dem.*

In numbers, let AC be √: 18 + √ 108; AB √: 18 − √ 108. then BC is √: 18 + √: 108 − √: 18 − √ 108.

d 17. 10.
e 11. *def.* 10.

PROP. LXXVIII.

D——E——F

If from a right line DF be taken away a right line DE, being incommensurable in power to the whole line DF, and with the whole DF making that which is composed of their squares medial, and the rectangle contained under the same lines rational, the line remaining EF is irrational, and is called a line making a whole space medial with a rational space.

For 2 FDE *a* is ṗv. *b* and DFq+DEq is μv. *c* therefore 2 FDE ⊏ DFq+DEq *d* (2 FDE+EFq) *e* therefore EF is ṗ. *Which was to be dem.*

In numbers, let DF be √: √ 216 + √ 72; DE √: √ 216 − √ 72. therefore EF is √: √ 216 + √ 72 − √: √ 216 − √ 72.

a *hyp. and sch.* 12. 10.
b *hyp.*
c *sch.* 12. 10.
d 7. 2.
e *sch.* 12. 10. & 11. *def.* 10.

PROP. LXXIX.

D EF

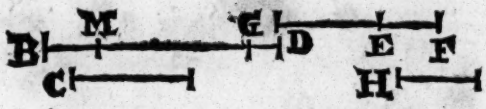
If from a right line DF be taken away a right line DE incommensurable in power to the whole DE, and which together with the whole makes that which is composed of their squares medial, and the rectangle contained under them, also medial and incommensurable to that which is composed of their squares, the remainder is irrational, and is called a line making a whole space medial with a medial space.

For 2 FDE, and FDq+DEq *a* are μα; *b* therefore EFq (*c* DFq+DEq − 2 FDE) is ṗv. *d* and so consequently EF is ṗ. *Which was to be dem.*

In numbers, let DF be √: √ 180 + √ 60. DE √: √ 180 − √ 60. then EF shall be √: √ 180 + √ 60 − √: √ 180 − √ 10.

a *hyp. and* 24. 10.
b 27. 10.
c *cor.* 7. 2.
d 11. *def.* 10.

Lemma.



If there be the ſame exceſs between the firſt magnitude BG, and the ſecond C (MG) as is between the third magnitude DF and the fourth H (EF;) then alternately, the ſame exceſs ſhall be between the firſt magnitude BG and the third DF, as is between the ſecond C and the fourth H.

a *hyp.* For becauſe that *a* to the equals BM, DE, are added the unequals MG, EF, that is, C, H;
b 15. *hx.* 1. the exceſs of the wholes BG, DF, *b* ſhall be equal to the exceſs of the parts added C, H. *Which was to be dem.*

Coroll.

Hence, Four magnitudes Arithmetically proportional, are alternately alſo Arithmetically proportional.

PROP. LXXX.

To an Apotome or reſidual line AB only one rational right line BC, being commenſurable in power only to the whole AB, is congruent, or can be joined.

a 22. 10. b 22. 10. c *cor.* 7. 2. d *lem.* 79. 10. e *hyp. and* 27. 10. f *ſch.* 12. 10. g 27. 10. If it be poſſible, let ſome other line BD be added to it; *a* then the rectangles ACB, ADB, *b* and ſo conſequently the double of them are *ua.* wherefore ſeeing ACq + BCq — 2 ACB *c* = ABq *c* = ADq + DBq — 2 ADB. therefore alternately AGq + BCq —: ADq + DBq *d* = 2 ACB —: 2 ADB. But ACq + BCq —: ADq + DBq *e* is *pv.* *f* therefore 2 ACB —: 2 ADB is *pv.* *Which is abſurd.*

PROP.

PROP. LXXXI.

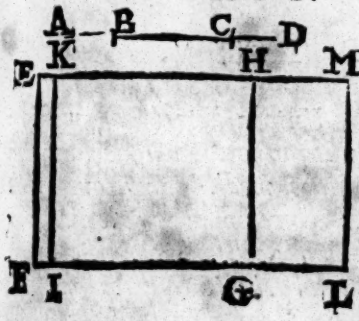
A B D C

To a first medial residual line AB only one medial right line BC, being commensurable only in power to the whole, and comprehending with the whole line a rational rectangle, can be joined.

Conceive BD to be such a line as may be joined to it; then because ACq and BCq, as well as ADq and BDq *a* are $\mu\alpha$ ▭. *b* also ACq + BCq, and ADq + BDq shall be $\mu\alpha$. *c* but the rectangles ACB, ADB, *d* and so 2 ACE and 2 ADB are $\rho\alpha$. *e* therefore 2 ACB —: 2 ADB, *f* that is, ACq + BCq —: ADq + BDq is $\rho\nu$. *g* *Which is absurd.*

a *hyp.* b 10. *and* 24. 10. c *hyp.* d *sch.* 12. 10. e *sch.* 27. 10. f 7. 2. *and lem.* 79. 10. g 27. 10.

PROP. LXXXII.



Unto a second medial residual line AB only one medial right line BC, commensurable only in power to the whole, and with it containing a medial rectangle, can be joined.

If it be possible, let some other line BD be added to it; and upon EF ρ, make the rectangle EG = ACq + BCq; as also the rectangle EL = ADq + BDq. likewise EI = ABq. Now 2 ACB + ABq = ACq + BCq = EG. therefore seeing EI = ABq, *a* also GK shall be = 2 ACB. moreover ACq and BCq *b* are $\mu\alpha$ ▭. *c* therefore EG (ACq + CBq) is $\mu\nu$. *d* therefore the breadth EH is ρ ▭ EF. *e* Further, the rectangle ACB *f* and so 2 ACB (KG) is $\mu\nu$. *d* therefore KH is also B, ρ EF. lastly, because ACq + BCq (EG) *g* ▭ 2 ACB (KG) and EG. KG *h* :: EH. KH. *k* therefore EH ▭ HK. *l* therefore EK is a residual line, whereto HK is congruent. by the same reason also

a 4. 2 *and* 3. *ax.* 1. b *hyp.* c 24. 10. d 23. 10. e *hyp.* f 24. 10. g *lem.* 26. 10. h 1. 6. k 10. 10. l 74. 10.

alſo ſhall KM be congruent to the ſaid EK. *which is repugnant to the* 80. *Prop. of this Book.*

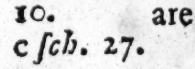
PROP. LXXXIII.

To a Minor line AB only one right line BC can be joined being incommenſurable in power to the whole: and making together with the whole line that which is compoſed of their ſquares rational, and the rectangle which is contained under them medial.

Conceive any other BD to be congruent to it; therefore whereas ACq -|- BCq, and ADq+ DBq *a* are $\dot{\rho}\alpha$, their exceſs (2 *b* ACB —: 2 ADB) *c* is $\dot{\rho}\nu$, *Which is abſurd*; becauſe ACB and ADB are $\mu\alpha$ by the Hyp.

a *hyp.*
b *lem.* 97. 10.
c *ſch.* 27. 10.
d 27. 10.

PROP. LXXXIV.



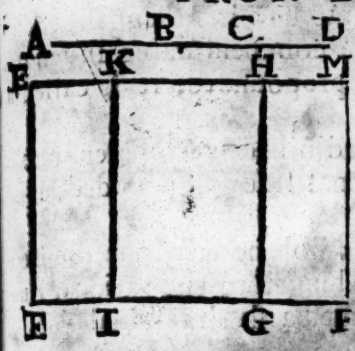
Unto a line (AB) making with a rational ſpace a whole ſpace medial only one right line BC can be joined, being incommenſurable in power to the whole, and making together with the whole that which is compoſed of their ſquares medial, and the rectangle which is contained under them rational.

Suppoſe ſome other BD to be congruent alſo to it; *a* then the rectangles ACB, ADB, *b* and ſo 2 ACB and 2 ADB are $\dot{\rho}\alpha$. therefore 2 ACB —: 2 ADB, *c* that is, ACq+BCq —: ADq+BDq *d* is $\dot{\rho}\nu$. *Which is abſurd.* ſince ACq -|- BCq, and ADq -|- BDq are $\mu\alpha$ by the Hyp.

a *hyp.*
b *ſch.* 12. 10.
c *lem.* 79. 10.
d *ſch.* 27. 10.

PROP.

PROP. LXXXV.



To a line AB, which with a medial ſpace makes a whole ſpace medial, can be joined only one right line BC, incommenſurable in power to the whole, and making with the whole both that which is compoſed of their ſquares medial, and the rectangle which is contained under them medial and incommenſurable to that which is compoſed of their ſquares.

Thoſe things being ſuppoſed which are done and ſhewn in the 82. Prop. of this Book; it is clear that EH and KH are ρ ▭ EF. Beſides, being that ACq+CBq, that is, the rectangle EG, *a* is ▭ ACB. *b* and ſo EG ▭ 2 ACB (KG;) and EG. KG *c* :: EH. KH; ſhall EH be ▭ KH. therefore EK is a reſidual line, and the line congruent to it is KH. In like manner may KM be ſhewn to be congruent to the ſaid reſidual EK, againſt the 80. Prop. of this Book.

a *hyp.* b 14. 10. c 1. 6.

Third Definitions.

A Rational line and a reſidual being propounded, if the whole be more in power than the line joined to the reſidual, by the ſquare of a right line commenſurable unto it in length; then

I. If the whole be commenſurable in length to the rational line propounded, it is called a firſt reſidual line.

II. But if the line adjoined be commenſurable in length to the rational line propounded, it is called a ſecond reſidual line.

III. If neither the whole nor the line adjoined be commenſurable in length to the rational line propounded; it is called a third reſidual line.

More-

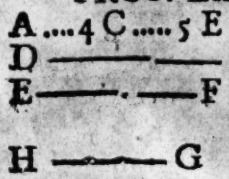
Moreover, if the whole be in more power than the line adjoined by the square of a right line incommensurable to it in length, then

IV. If the whole be commensurable in length to the rational line propounded, it is called a fourth residual line.

V. But if the line adjoined be commensurable in length to the rational line propounded, it is a fifth residual.

VI. If neither the whole nor the line adjoined be commensurable in length to the rational line propounded, it is termed a sixth residual line.

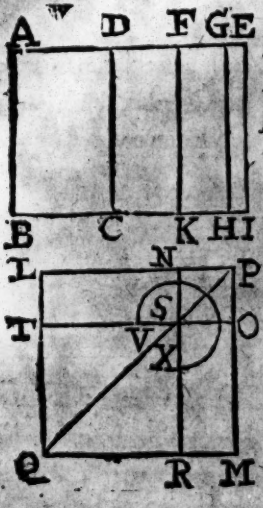
PROP. LXXXVI, 87, 88, 89, 90, 91.



To find out a first, second, third, fourth, fifth, and sixth residual line.

Residual lines are found out by subducting the less names or parts of binomials from the greater. *ex. gr.* Let $6 + \sqrt{20}$ be a first binom. then shall $6 - \sqrt{20}$ be a first residual. So that it is not necessary to repeat more concerning the finding of them out.

Lemma.



Let AC be a rectangle contained under the right lines AB, AD. Let AD be drawn forth to E, and DE equally divided in F. and let the rectangle AGE be = FEq. and the rectangles AI, DK, FH, be finished. Then let the square LM = AH be made, and the square NO = GI; and the lines NSR, OST, produced.

I say 1. the rectangle AI = LM + NO = TOq + SOq. which appears by the constr.

2. The

2. *The rectangle DK = LG.* For becauſe the rectangle AGE *a* = FEq. *b* thence are AG, FE, GE, ∴ *c* and ſo AH, FI, GI ∴, *a* that is, LM, FI, NO ∴; but LM, LO, NO *d* are ∴; therefore FI = *e* LO *f* = DK = *g* NM.

3. *Hence, AC = AI − DK − FI = LM + NO − LO − NM = TR.*

4. *h It is manifeſt that DF, FE, DE, are* □.

5. *If AE* ⊥ *DE, and AE* □ √ *AEq − DEq, k then ſhall AG, GE, AE be* □.

6. *Alſo, becauſe AE l* □ *DE, m thence ſhall AE, FE be* □. *n and ſo AI, FI, that is, LM | NO and LO are* □.

7. *Becauſe AG* * □ *GE, n ſhall AH, GI, that is, LM, NO be* □.

8. *But becauſe AE l* □ *DE, o therefore ſhall* * *before FE, GE be* □, *n and ſo the rectangle FI* □ *GI, that is, LO* □ *NO. wherefore ſeeing LO. NO p :: TS. SO. q therefore ſhall TS, SO be* □.

9. *If AE be put* □ √ *AEq − DEq, r then ſhall AG, GE, AE be* □.

10. *ſ Wherefore the rectangles AH, GI, that is, TOq, SOq ſhall be* □.ſ

a *conſtr.*
b 17. 6.
c 1. 6.
d *ſch.* 2. 6.
e 9. 5.
f 36. 1.
g 43. 1.
h 16. 10.
k 18. *and* 10. 10.
l *hyp.*
m 13. 10.
n 1. 6. *and* 10. 10.
o 14. 10.
p 2. 6.
q 10. 10.
r 19. 10. *&* 17. 10.
ſ 1. 6. *and* 10. 10.

PROP.

PROP. XCII.

If a ſpace AC be contained under a rational line AB, and a firſt reſidual line AD (AE — DE) the right line TS, which contains the ſpace AC in power, is a reſidual line.

Uſe the foregoing *Lemma* for a preparatory to the demonſtration of this Prop. Therefore TS = √ AC. Alſo AG, GE, AE, are □; therefore ſince AE □ *a* AB ρ̂, *b* alſo AG and GE ſhall be □ AB. *c* therefore the rectangles AH and GI, that is, TOq and SOq are ρ̂α. *d* Likewiſe TO, SO, are ρ̂ ⊓. *e* and conſequently TS is a reſidual line. *Which was to be dem.*

a *hyp.*
b 12. 10.
c 20. 10.
d *lem.* 91. 10.
e 74. 10.

PROP. XCIII.

See the preceding Scheme.

If a ſpace AC be contained under a rational line AB and a ſecond reſidual AD (AE — DE) the right line TS, containing the ſpace AC in power, is a firſt medial reſidual line.

Again, by the foregoing *Lemma*, AG, GE, AE are □. therefore *a* ſince AE is ρ̂ □ AB, *b* alſo AG, GE, ſhall be ρ̂ □ AB. *c* therefore the rectangles AH, GI, that is, TOq, SOq are μα. *d* likewiſe TO ⊓ SO. Laſtly, becauſe DE *e* □ AB ρ̂, *f* the right angle DI, and the half thereof DK or LO, that is, TOS ſhall be ρ̂ν. *g* from whence it follows that TS (√ AC) is a firſt medial reſidual. *Which was to be dem.*

a *hyp.*
b 13. 10.
c 22. 10.
d *lem.* 74. 10.
e *hyp.*
f 20. 10.
g 75. 10.

PROP.

PROP. XCIV.

See Scheme 92.

If a ſpace AC be contained under a rational line AB and a third reſidual AD (AE — DE) the right line TS containing in power the ſpace AC is a ſecond medial reſidual line.

As in the former, TO and SO are μ. Therefore becauſe DE *a* is $\dot{\rho}$ ⊥ AB. *b* the rectangle DI, *c* and ſo DK, or TOS, ſhall be $\mu\nu$. *d* therefore TS $= \sqrt{}$ AC is a ſecond medial reſidual. *Which was to be dem.*

a *hyp.*
b 21. 10.
c 24. 10.
d 76. 10.

PROP. XCV.

See Scheme 92.

If a ſpace AC be contained under a rational line AB and a fourth reſidual AD (AE — DE) the right line TS containing the ſpace AC in power, is a Minor line.

As before, TO *a* ⊥ SO. Therefore becauſe AE *b* is $\dot{\rho}$ ⊥ AB, *c* ſhall AI (TOq + SOq) be $\dot{\rho}\nu$. but, as before, the rectangle TOS is $\mu\nu$. *d* therefore TS $= \sqrt{}$ AC is a Minor line. *Which was to be dem.*

a *lem.* 91.
10.
b *hyp.*
c 20. 10.
d 77. 10.

PROP. XCVI.

See Scheme 92.

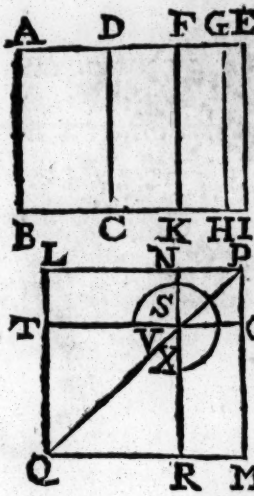
If a ſpace AC be contained under a rational line AB and a fifth reſidual AD (AE — DE) the right line TS containing in power the ſpace AC, is a line which makes with a rational ſpace the whole ſpace medial.

For again TO ⊥ SO. therefore ſince AE *a* is $\dot{\rho}$ ⊥ AB. *b* alſo AI, that is, TOq + SOq, ſhall be $\mu\nu$. But, as in the 93. the rectangle TOS is $\dot{\rho}\nu$. *c* whence TS $= \sqrt{}$ AC is a line which with $\dot{\rho}\nu$ makes a whole $\mu\nu$. *Which was to be dem.*

a *hyp.*
b 22. 10.
c 78. 10.

PROP.

PROP. XCVII.

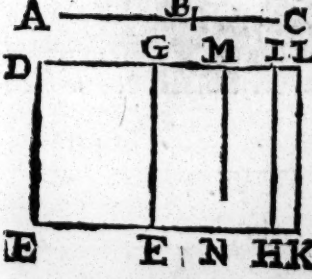


If a ſpace AC be contained under a rational line AB, and a ſixth reſidual AD (AE − DE) the right line TS containing in power the ſpace AC, is a line making with a medial rectangle, a whole ſpace medial.

As often above, TO ⊓ SO. alſo, as in 96. TOq + SOq is $\mu\nu$. but the rectangle TOS is $\rho\nu$, as in 94. [a] Laſtly, TOq + SOq ⊓ TOS. [b] therefore TS = √ AC is a line which with $\mu\nu$ makes a whole $\mu\nu$. *Which was to be dem.*

a *lem.* 91. 20.
b 79. 10.

Lemma.



* *cor.* 16. 6. *Upon a right line DE* * *apply the rectangles DF = ABq, and DH = ACq, and IK = BCq. and let GL be biſected in M, and the line MN drawn parallel to GF.*

Then 1. *The rectangle DK is = ACq + BCq, as the conſtruction manifeſts.*

2. *The rectangle* ACB = GN or MK. For DK [a] = ACq + BCq [b] = 2 ACB + ABq. but ABq [c] = DF. therefore GK [c] = 2 ACB. and [d] conſequently GN or MK = ACB.

3. *The rectangle* DIL = MLq. For becauſe ACq. ACB [e] :: ACB. BCq, that is, DH, MK :: MK. IK. [e] thence is DI. ML :: ML. IL. [f] therefore DIL = MLq.

a *conſtr.*
b 7. 2.
c 3. *ax.* 1.
d 7. *ax.* 1.
e 1. 6.
f 17. 6.

4. *If*

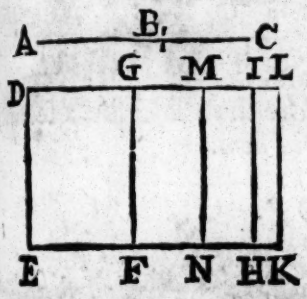
4. *If AC be taken* ⊐ *BC, then DK ſhall be* □ ACq. For ACq + BCq (DK) *g* □ ACq. g 16. 10.

5. *Likewiſe* DL □ √ DLq − GLq. For becauſe DH (ACq) □ IK (BCq) *h* thence ſhall DI be □ IL. *k* therefore √ DLq − GLq □ DL. h 10. 10. k 18. 10.

6. *Alſo* DL □ GL. For ACq + BCq □ 2 ACB. that is, DK □ GK. *m* therefore DL □ GL. *l* 1 *lem.* 26. 10. m 10. 10.

7. *But if AC be taken* ⊏ *BC, n then DL ſhall be* □ √ *DLq − GLq.* n 19. 10.

PROP. XCVIII.



The ſquare of a reſidual line AB (AC − BC) applied to a rational line DE, makes the breadth DG a firſt reſidual line.

Do as is injoined in the *Lemma* next preceding. Then becauſe AC, BC, *a* are ρ̊ ⊐. *b* alſo DK (ACq + BCq) ſhall be □ ACq. *c* therefore DK is ρ̊ν. *d* wherefore DL is ρ̊ □ DE. *e* Likewiſe the rectangle GK (2 ACB) is μν. therefore GL is ρ̊ □ DE. *g* and conſequently DL □ GL. *h* But DLq □ GLq. *k* therefore DG is a reſidual, *l* and that of the firſt order (becauſe *m* AC ⊐ BC, and therefore DL □ √ DLq − GLq.) *Which was to be dem.*

a *hyp.* b *lem.* 97. 10. c *ſch.* 12. 10. d 21. 10. e 22. *and* f 24. 10. f 23. 10. g 13. 10. h *ſch.* 12. 10. k 74. 10. l 1. *def.* 85. 10. m *lem.* 97. 10.

PROP.

PROP. XCIX.

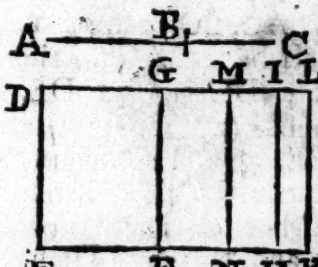
See the following Scheme.

The ſquare of a firſt medial reſidual line AB (AC — BC) applied to a rational line DE, makes the breadth DG a ſecond reſidual line.

Suppoſing the foregoing *Lemma*; becauſe AC and BC *a* are μ. $\square$, *b* thence ſhall DK (ACq + BCq) be $\square$ ACq. *c* wherefore DK is $\mu\nu$. *d* therefore DL is ρ $\square$ DE. *e* alſo GK (2 ACB) is $\rho\nu$. *f* therefore GL is ρ $\square$ DE, *g* wherefore DL $\square$ GL. *h* But DLq $\square$ GLq. *k* therefore DG is a reſidual line: and becauſe DL is $\square\sqrt{}$ DLq — GLq. *m* therefore ſhall DG be a ſecond reſidual. *Which was to be dem.*

a *hyp.*
b *lem.* 97. 10.
c 24. 10.
d 23. 10.
e *hyp. and ſch.* 12. 10.
f 21. 19.
g 13. 10.
h *ſch.* 12. 10.
k 74. 10.
l *lem.* 97. 10.
m 2. *def.* 85. 10.

PROP. C.



The ſquare of a ſecond medial reſidual line AB (AC — BC) applied to a rational line DE makes the breadth DG a third reſidual line.

Again DK is $\mu\nu$. *a* wherefore DL is ρ $\square$ DE. alſo DGK is $\mu\nu$. *b* whence GL is ρ $\square$ DE. *c* likewiſe DK $\square$ GK. *c* wherefore DL $\square$ GL. *d* but DLq $\square$ GLq. *e* therefore DG is a reſidual line, and that of *f* the third order, *g* becauſe DL $\square\sqrt{}$ DLq — GLq. *Which was to be dem.*

a 23. 10.
b *lem.* 26. 10.
c 1. 6. *and* 10. 10.
d *ſch.* 12. 10.
e 74. 10.
f 3. *def.* 85. 10.
g *lem.* 97. 10.

PROP. CI.

See the foregoing Scheme.

The ſquare of a Minor line AB (AC — BC) applied

to

to a rational line DE, makes the breadth DG a fourth reſidual.

As before, ACq + BCq, that is, DK is $\dot{\rho}\nu$. *a* therefore DL is $\dot{\rho}$ □ DE. but the rectangle ACB,* and ſo GK (2 ACB) * is $\mu\nu$. *b* wherefore GL is $\dot{\rho}$ □ DE, *c* therefore DL □ GL. *b* but *c* DLq □ GLq. and becauſe * ACq □ BCq, *e* thence ſhall DL be □ √ DLq — GLq. *f* therefore DG has the conditions required to a fourth reſidual. *Which was to be dem.*

a 21. 10.
* *hyp.*
b 23. 10.
c 13. 10.
d *ſch.* 12. 10.
e *lem.* 97. 10.
f 4. *def.* 85. 10.

PROP. CII.

See Scheme 100.

The ſquare of a line AB (AC — BC) which makes with a rational ſpace the whole ſpace medial, applied to a rational line DE, makes the breadth DG a fifth reſidual line.

For, as above, DK is $\mu\nu$. *a* wherefore DL is $\dot{\rho}$ □ DE. alſo GK is $\dot{\rho}\nu$. *b* whence GL is $\dot{\rho}$ □ DE. *c* therefore DL □ GL. *d* but DLq □ GLq. Moreover DL *e* □ √ DLq — GLq. wherefore DG *f* is a fifth reſidual. *Which was to be dem.*

a 23. 10.
b 21. 10.
c 13. 10.
d *ſch.* 12. 10.
e *lem.* 97. 10.
f 5. *def.* 85. 10.

PROP. CIII.

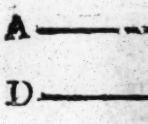
See the laſt Scheme.

The ſquare of a line AB (AC — BC) making with a medial ſpace the whole ſpace medial, applied to a rational line DE, makes the breadth DG a ſixth reſidual line.

As above DK and GK are $\mu\alpha$; *a* wherefore DL and GL are $\dot{\rho}$ □ DE. alſo DK *b* □ GK. *c* whence DL □ GL. *d* therefore DG is a reſidual. *b* And whereas ACq □ BCq. and ſo DL □ √ DLq — GLq, *e* therefore DG ſhall be a ſixth reſidual. *Which was to be dem.*

a 23. 10.
b *hyp.* and *lem.* 97. 10.
c 10. 10.
d 74. 10.
e 6. *def.* 85. 10.

PROP. CIV.



A right line DE commenſurable in length to a reſidual AB (AC — BC) is it ſelf alſo a reſidual, and of the ſame order.

Lemma.

Let AB. DE :: AC. DF. *and* AB ▱ DE.

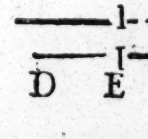
I ſay AC -|- BC ▱ DF + EF. For AC. BC *a* :: DF. EF, therefore by addition AC + BC. BC :: DF + EF. FE, therefore by inverſion AC + BC. DF -|- EF :: BC. EF. *a* but BC ▱ EF. *b* therefore AC -|- BC ▱ DF -|- EF. *Which was to be dem.*

a *lem.* 66. 10.
b 10. 10.

a 12. 6. *a* Make AB. DE :: AC. DF. *b* therefore AC + BC ▱ DF + EF. therefore ſeeing AC -|- BC *c* is a binomial, *d* DF -|- EF ſhall be a binomial too, and of the ſame order. *e* wherefore DF - EF is a reſidual of the ſame order with AC - BC. *Which was to be dem.*

b *lem.* 103. 10.
c *hyp.*
d 67. 10.
e *by def.* 85. 10.

PROP. CV.



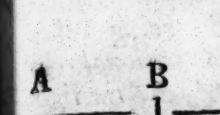
A right line DE commenſurable to a medial reſidual line AB (AC — BC) is it ſelf a medial reſidual, and of the ſame order.

a 12. 6. Again *a* make AB. DE :: AC. DF. *b* whence AC + BC ▱ DF + EF. *c* therefore DF + EF is a bimedial of the ſame order with AC + BC, *d* and conſequently DF — EF ſhall be a medial reſidual of the ſame order with AC — BC. *Which was to be demonſtrated.*

b *lem.* 103. 10.
c 68. 10.
d 75. *and* 76. 10.

PROP.

PROP. CVI.

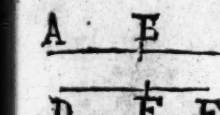


A right line DE commenſurable to a Minor line AB (AC — BC) is it ſelf alſo a Minor line.

Make AB. DE :: AC. DF. *a* then is AC + BC ▭ DF + EF. But AC + BC *b* is a Major line; *c* therefore DF + EF is alſo a Major line; *d* and conſequently DF — EF is a Minor line. *Which was to be dem.*

a *lem.* 103. 10.
b *hyp.*
c 69. 10.
d 77. 10.

PROP. CVII.

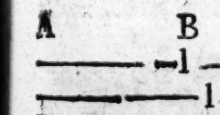


A right line DE commenſurable to a line AB (AC — BC) which makes with a rational ſpace the whole ſpace medial, is it ſelf alſo a line making with a rational ſpace the whole ſpace medial.

For, accordingly as in the former, we may ſhew DF + EF to contain in power ρν and μν. *a* whence DF — EF is a line making, *&c.*

a 78. 10.

PROP. CVIII.

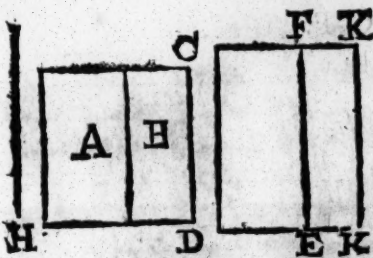


A right line DE commenſurable to a line AB (AC — BC) which with a medial ſpace makes the whole ſpace medial, is it ſelf a line making with a medial ſpace the whole ſpace medial.

For according to the preceding DF + EF ſhall contain in power 2 μα. *a* therefore DF — EF ſhall be, as in the Prop.

a 79. 10.

PROP. CIX.



A medial rectangle B being taken from a rational rectangle A+B, the right line H which contains in power the ſpace remaining A, is one of thoſe two irrational lines, viz. either a reſidual line, or a Minor line.

Upon CD $\dot{\rho}$ make the rectangles CI = A+B. and FI = B. whence CE *a* = A = Hq. wherefore becauſe CI *b* is $\dot{\rho}v$. *c* therefore CK is $\dot{\rho}$ ⊓ CD. but being FI *b* is μv, *d* ſhall FK be $\dot{\rho}$ ⊓ CD. *e* whence CK ⊓ FK. *f* therefore CF is a reſidual line. Wherefore if CK be ⊓ $\sqrt{}$ CKq − FKq, *g* then CF ſhall be a firſt reſidual, *h* therefore $\sqrt{}$ CE (H) is a reſidual line. But if CK ⊓ $\sqrt{}$ CKq − FKq. *k* then CF ſhall be a fifth reſidual; and conſequently H ($\sqrt{}$ CE) *l* ſhall be a Minor line. *Which was to be dem.*

a 5. *ax.* 1.
b *hyp. and conſtr.*
c 21. 10.
d 23. 10.
e 13. 10.
f 74. 10.
g 1. *def.* 85. 10.
h 92. 10.
k 4. *def.* 85. 10.
l 95. 10.

PROP. CX.

See the prec. Scheme.

A rational rectangle B being taken away from a medial rectangle A+B, other two irrational lines are made, namely either a firſt medial reſidual line, or a line making with a rational ſpace the whole ſpace medial.

Upon CD the propounded $\dot{\rho}$ make the rectangles CI = A+B, and FI = B. *a* whence CE = A = Hq. Therefore becauſe CI *b* is μv, *c* ſhall CK be $\dot{\rho}$ ⊓ CD. but becauſe FI *b* is $\dot{\rho}v$, *d* thence FK $\dot{\rho}$ ⊓ CD. *e* whence CK ⊓ FK. *f* therefore CF is a reſidual, *g* and that a ſecond. If CK ⊓ $\sqrt{}$ CKq − FKq, *h* then H ($\sqrt{}$ CE) is a firſt medial reſidual. But if CK ⊓ $\sqrt{}$ CKq − FKq. *k* then ſhall CF be a fifth reſidual; and *l* conſequently H ($\sqrt{}$ CE) ſhall be a line making μv with $\dot{\rho}v$. *Which was to be dem.*

a 3. *ax.* 1.
b *hyp. and conſtr.*
c 23. 10.
d 21. 10.
e 13. 10.
f 74. 10.
g 2. *def.* 85. 10.
h 93. 10.
k 5. *def.* 85. 10.
l 96. 10.

PROP.

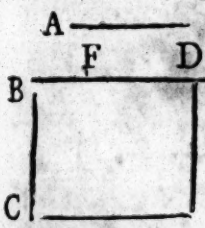
PROP. CXI.

See the ſame Scheme.

A medial ſpace B being taken away from a medial ſpace A+B, which is incommenſurable to the whole A+B, the other two irrational lines are made, viz. either a ſecond medial reſidual line, or a line making with a medial ſpace the whole ſpace medial.

Upon CD $\dot{\rho}$ make the rectangles CI = A + B, a 3. *ax.* 1.
and FI = B. *a* wherefore CE = A = Hq. Becauſe b 23. 10.
therefore CI is $\mu\nu$, *b* thence CK is $\dot{\rho}$ ⊓ CD, c *hyp.*
and in like manner FK $\dot{\rho}$ ⊓ CD. Likewiſe be- d 10. 10.
cauſe CI *c* ⊓ FI, *d* therefore CK ⊓ FK. *e* e 74. 10.
wherefore CF is a reſidual, *f* namely a third. If f 3. *def.* 85.
CK ⊓ √ CKq − FKq, *g* whence H (√ CE ſhall 10.
be a ſecond medial reſidual, but if CK ⊓ √ g 94. 10.
CKq − FKq *h* then ſhall CF be a ſixth reſidual. h 6. *def.* 85.
k wherefore A ſhall be a line making $\mu\nu$ with μ. 10.
Which was to be dem. k 97. 10.

PROP. CXII.



A reſidual line A is not the ſame with a bimedial line.

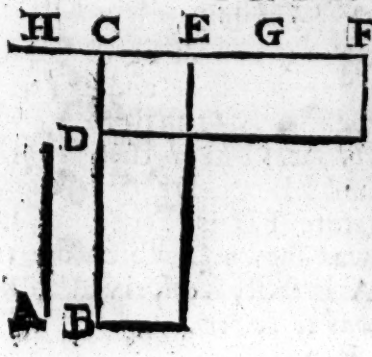
Upon BC propounded $\dot{\rho}$ make the rectangle CD = Aq. Therefore ſeeing A a 98. 10.
is a reſidual, *a* BD ſhall be a firſt reſidual, to which let DE be the line con- b 74. 10.
gruent, or that may be adjoined. *b* wherefore BE, c 1. *def.* 85.
DE, are $\dot{\rho}$ ⊐. *c* and BE ⊓ BC. If you con- 10.
ceive A to be a binomial, then BD is a firſt bin, whoſe names let be BF, FD; and let BF be ⊏ d 37. 10.
FD. *d* therefore BF, FD are $\dot{\rho}$ ⊐; and BF *e* e 1. *def.* 48.
⊓ BC. therefore ſince BC ⊓ BE *f* ſhall BE be 10.
⊓ BF. *g* and thence BE ⊓ FE. *h* therefore f 12. 10.
FE is $\dot{\rho}$. Likewiſe becauſe BE ⊓ DE, *k* ſhall g *cor.* 16. 10
FE be ⊓ DE. *l* wherefore FD is a reſidual, h *ſch.* 12.
and ſo FD is $\dot{\rho}$. but it was ſhewn $\dot{\rho}$, which are 10.
repugnant. Therefore A is falſly conceived to k 14. 10.
be a binomial. *Which was to be dem.* l 74. 10.

The names of the 13 *irrational lines differing one from another.*

1. A Medial line.
2. A binomial line; of which there are ſix ſpecies.
3. A firſt bimedial line.
4. A ſecond bimedial.
5. A Major line.
6. A line containing in power a rational ſuperficies, and a medial ſuperficies.
7. A line containing in power two medial ſuperficies.
8. A reſidual line; of which there are alſo 6 kinds.
9. A firſt medial reſidual line.
10. A ſecond medial reſidual line.
11. A Minor line.
12. A line making with a rational ſuperficies the whole ſuperficies medial.
13. A line making with a medial ſuperficies the whole ſuperficies medial.

Being the differences of breadths do argue differences of right lines, whoſe ſquares are applied to ſome rational line, and it is demonſtrated in the prec. Propoſitions that the breadths which ariſe from applying of the ſquares of theſe 13 *lines do differ one from another, it evidently follows that theſe* 13 *lines do alſo differ one from another.*

PROP. CXIII.

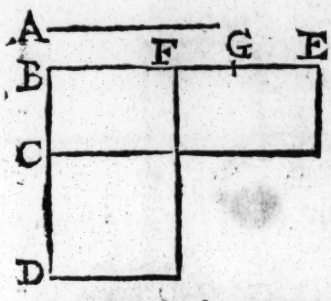


The ſquare of a rational line A applied to a binomial BC (BD + DC) makes the breadth EC a reſidual line, whoſe names EH, CH, are commenſurable to the names BD, DO, of the binomial line, and in

in the ſame proportion (EH. BD :: CH. DC.) and moreover the reſidual line EC which is made, is of the ſame order with BC the binomial.

Upon DC the leſs name *a* make the rectangle DF = Aq = BE. whence BC. CD *b* :: FC. CE. therefore by diviſion, BD. DC :: FE. EC. And whereas BD *c* ⊏ DC, *d* thence FE ſhall be ⊏ EC. Take EG = EC, and make FG. GE :: EC. CH. Then EH, and CH ſhall be the names of the reſidual EC, whereunto all is agreeable that is propounded in the Theorem. For being that by addition FE. GE (EC) :: EH. CH, therefore FH, EH *e* :: EH. CH *f* :: FE. EC *f* :: BD. BC, wherefore ſince BD *g* ⊐ DC. *h* thence ſhall EH be ⊐ CH. *h* and FHq ▭ EHq. Therefore cauſe FHq. EHq :: FH. CH. *h* ſhall FH be ▭ CH, *l* and ſo FC ▭ CH. Moreover CD *g* is $\dot{\rho}$, and DF (Aq) *g* is $\dot{\rho}v$. *m* therefore FC is $\dot{\rho}$, ▭ CD. whence alſo CH is $\dot{\rho}$ ▭ CD. *n* therefore EH, CH are $\dot{\rho}$ and ⊐. as before, *o* therefore EC is a reſidual line, to which CH may be joined. Furthermore EH. CH *f* :: BD. DC. and ſo by inverſion EH. BD :: CH. DC. whence becauſe CH *f* ▭ DC, *p* ſhall EH be ▭ BD. But ſuppoſe BD ▭ $\sqrt{BDq-DCq}$. *q* then ſhall EH be ▭ $\sqrt{EHq-CHq}$. Alſo if BD ▭ $\dot{\rho}$ propounded, then ſhall EH be ▭ to the ſame $\dot{\rho}$. *f* that is, if BC be a firſt binomial, *t* EC ſhall be a firſt reſidual. In like manner, if DC be to the ▭ propounded $\dot{\rho}$. *r* then is CH ▭ to the ſame $\dot{\rho}$. *u* that is, if BC be a ſecond binomial, *x* EC ſhall be a ſecond reſidual; and if this be a third binom. then that ſhall be a third reſidual, *&c.* But if BD be ▭ $\sqrt{BDq - DCq}$, *y* then ſhall EH be ▭ $\sqrt{EHq - CHq}$. therefore if BC be a 4th, 5th, or 6th binomial, EG ſhall be likewiſe a 4th, 5th or 6th reſidual. *Which was to be dem.*

PROP. CXIV.

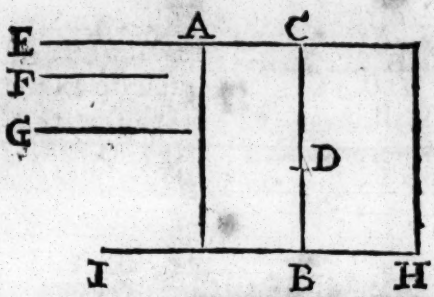


The ſquare of a rational line A applied to a reſidual line B C (BD − CD) makes the breadth BE a binomial; whoſe names BE, GE are commenſurable to the names BD, BC of the reſidual line BC, and in the ſame proportion, and moreover, the binomial line which is made (BE) is of the ſame order with the reſidual line (BC.)

a *cor.* 16. 6. *a* Make the rectangle DF = Aq. and BF. FE
b 12. 6. *b* :: EG. GF. whence for that DF = Aq = CE, *c*
c 14. 6. therefore BD. BC :: BE. BF, therefore by con-
d 19. 5. verſion of proportion BD. CD :: BE. FE :: EG.
e *hyp.* GF :: *d* BG. EG. but BD *e* ⊓ CD. *f* therefore
f 10. 10. BG ⊓ GE. therefore becauſe BGq. GEq *g* ::
g *cor.* 20. 6. BG. GF. *b* ſhall BG be ⊓ GF. *k* and ſo BG
h 10. 10. ⊓ BF. moreover BD *e* is $\dot{\rho}$, and the rectangle
k *cor.* 16. DF (Aq) *e* is $\dot{\rho}\rho$. *l* therefore BF is $\dot{\rho}$ ⊓ BD. *m*
10. therefore alſo BG is $\dot{\rho}$ ⊓ BD. *n* therefore BG,
l 21. 10. GE are $\dot{\rho}$ ⊓. *e* wherefore BE is a binomial.
m 12. 10. Laſtly, becauſe BD. CD :: BG. GE. and inverſe-
n *ſch.* 12. ly BD. BG :: CD. GE. and BD ⊓ BG. *p* thence
10. ſhall CD be ⊓ GE. therefore if CB be a firſt
o 37. 10. reſidual, BE ſhall be a firſt binomial, *&c.* as in
p 10. 10. the prec. therefore, *&c.*

PROP.

PROP. CXV.



If a ſpace AB be contained under a reſidual line AC (CE — AE) and a binomial CB, whoſe names CD, DB are commenſurable to the names CE, AE, of the reſidual line, and in the ſame proportion (CE. AE :: CD. DB) then the right line F which contains in power that ſpace AB, is irrational.

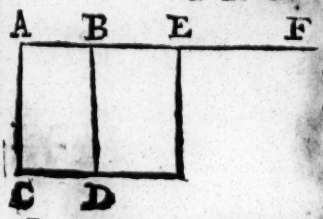
Let G be $\dot{\rho}$. and make the rectangle CH = Gq; *a* then ſhall BH (HI — IB) be a reſidual line, and HI *a* ⊓ CD *b* ⊓ CE. *a* and BI ⊓ DB *a* and HI.BI :: CD.DB *b* :: CE.EA. therefore by inverſion HI. CE :: BI. EA. *c* therefore BH. AC :: HI. CE :: BI. EA. *c* therefore BH.AC :: HI. CE :: BI. EA. wherefore ſince *d* HI ⊓ CE, *e* thence BH ⊓ AC. *f* therefore the rectangle HC ⊓ BA. But HC (Gq) *b* is $\dot{\rho}v$, *g* therefore BA (Fq) is $\dot{\rho}v$: and conſequently F is $\dot{\rho}$. *Which was to be dem.*

a 113. 10. b *hyp.* c 19. 5. d 12. 10. e 10. 10. f 1. 6. *and* g *ſch.* 12. 10.

Coroll.

Hereby it appears that a rational ſuperficies may be contained under two irrational right lines.

PROP. CXVI.

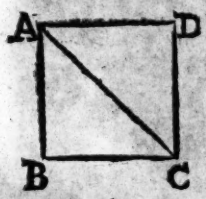


Of a medial line AB are produced infinite irrational lines BE, EF, &c. whereof none is of the ſame kind with any of the precedent.

Let AC be propounded $\dot{\rho}$. and AD a rectangle con-

contained under AC, AB. *a* therefore AD is $\dot{\rho}v$.
a *lem.* 38. 10. Take BE $= \sqrt{}$ AD. *b* then BE is $\dot{\rho}$, and the
b 11. 10. same with none of the former. For no square of any of the former being applied to $\dot{\rho}$, makes the breadth medial. Let the rectangle DE be finished, *a* then DE shall be $\dot{\rho}v$. and *b* consequently EF ($\sqrt{}$ DE) shall be $\dot{\rho}$, and not the same with any of the former, for no square of the former being applied to $\dot{\rho}$, makes the breadth BE. therefore, &c.

PROP. CXVII.



Let it be required to shew that in square figures BD. the diameter AC is incommensurable in length to the side AB.

a 47. 1. For ACq. ABq *a* :: 2. 1 *b* ::
b *cor.* 24. 8. not Q. Q. *c* therefore AC ⊔
c 9. 10. AB. *Which was to be dem.* This Theorem was of great note with the ancient Philosophers; so that he that understood it not was esteemed by *Plato* undeserving the name of a Man, but rather to be reckoned among Brutes.

The End of the tenth Book.

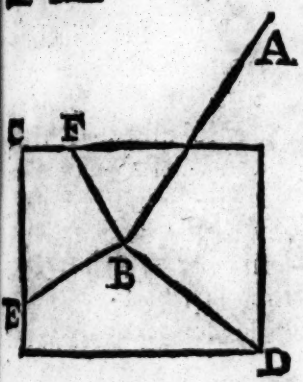
THE

THE ELEVENTH BOOK OF EUCLIDE's ELEMENTS.

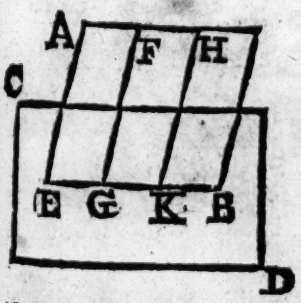
Definitions.

A Solid is that which hath length, breadth, and thickneſs.

II. The term, or extreme of a ſolid is a Superficies.

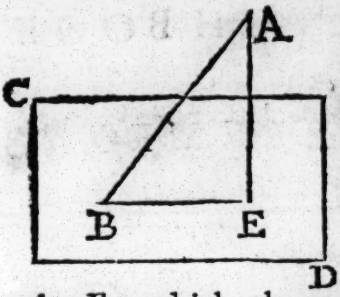


III. A right line AB is perpendicular to a Plane CD, when it makes right angles ABD, ABE, ABF, with all the right lines BD, BE, BF, that touch it, and are drawn in the ſaid Plane.

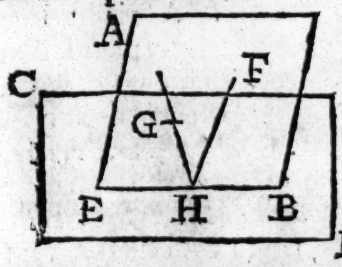


IV. A Plane AB, is perpendicular to a Plane CD, when the right lines FG, HK, drawn in one Plane AB to the line of common ſection of the two Planes EB, and making right angles therewith, do alſo make right angles with the other Plane CD.

V. The



V. The inclination of a right line AB to a Plane CD, is, when a perpendicular AE is drawn from A the higheſt point of that line AB to the plane CD, and another line EB drawn from the point E, which the perpendicular AE makes in the Plane CD, to the end B of the ſaid line AB which is in the ſame Plane, whereby the angle is acute ABE which is contained under the inſiſting line AB, and the line drawn in the plane EB.



VI. The inclination of a Plane AB, to a Plane CD, is an acute angle FHG contained under the right lines FH, GH, which being drawn in either of the Planes AB, CD to the ſame point H of the common ſection BE, make right angles FHB, GHB, with the common ſection BE.

VII. Planes are ſaid to be inclined to other Planes in the ſame manner, when the ſaid angles of inclination are equal one to another.

VIII. Parallel Planes are thoſe which being prolonged never meet.

IX. Like ſolid figures are ſuch as are contained under like Planes equal in number.

X. Equal and like ſolid figures are ſuch as are contained under like Planes equal both in multitude and magnitude.

XI. A ſolid angle is the inclination of more than two right lines which touch one another, and are not in the ſame ſuperficies.

Or

Or thus;

A ſolid angle is that which is contained under more than two plane angles not being in the ſame ſuperficies, but conſiſting all at one point.

XII. A Pyramide is a ſolid figure comprehended under divers planes ſet upon one plane, (which is the baſe of the Pyramide) and gathered together to one point.

XIII. A Priſme is a ſolid figure contained under planes, whereof the two oppoſite are equal, like, and parallel; but the others are parallelograms.

XIV. A Sphere is a ſolid figure made when the diameter of a circle abiding unmoved, the ſemicircle is turned round about, till it return to the ſame place from whence it began to be moved.

Coroll.

Hence, all the rayes drawn from the center to the ſuperficies of a ſphere, are equal amongſt themſelves.

XV. The Axis of a ſphere, is that fixed right line, about which the ſemicircle is moved.

XVI. The Center of a ſphere, is the ſame point with that of the ſemicircle.

XVII. The Diameter of a ſphere, is a right line drawn thro' the center, and terminated on either ſide in the ſuperficies of the ſphere.

XVIII. A Cone is a figure made, when one ſide of a rectangled triangle (*viz.* one of thoſe that contain the right angle) remaining fixed, the triangle is turned round about till it return to the place from whence it firſt moved. And if the fixed right line be equal to the other which contains the right angle, then the Cone is a rectangled Cone: but if it be leſs, it is an obtuſe-angled Cone; if greater an acute-angled Cone.

XIX. The Axis of a Cone is that fix'd line about which the triangle is moved.

XX. The

XX. The Baſe of a Cone is the circle, which is deſcribed by the right line moved about.

XXI. A Cylinder is a figure made by the moving round of a right-angled parallelogram, one of the ſides thereof, (namely which contain the right angle) abiding fix'd, till the parallelogram be turned about to the ſame place, where it began to move.

XXII. The Axis of a Cylinder is that quieſcent right line, about which the parallelogr. is turned.

XXIII. And the Baſes of a Cylinder are the circles which are deſcribed by the two oppoſite ſides in their motion.

XXIV. Like Cones and Cylinders, are they, both whoſe Axes and Diameters of their Baſes are proportional.

XXV. A Cube is a ſolid figure contained under ſix equal ſquares.

XXVI. A Tetraedron is a ſolid figure contained under four equal and equilateral triangles.

XXVII. An Octaedron is a ſolid figure contained under eight equal and equilateral triangles.

XXVIII. A Dodecaedron is a ſolid figure contained under twelve equal, equilateral and equiangular Pentagones.

XXIX. An Icoſaedron is a ſolid figure contained under twenty equal and equilateral triangles.

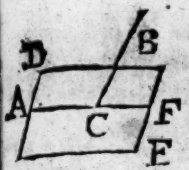
XXX. A Parallelepipedon is a ſolid figure contained under ſix quadrilateral figures, whereof thoſe which are oppoſite are parallel.

XXXI. A ſolid figure is ſaid to be inſcribed in a ſolid figure, when all the angles of the figure inſcribed are comprehended either within the angles, or in the ſides, or in the planes of the figure wherein it is inſcribed.

XXXII. Likewiſe a ſolid figure is then ſaid to be circumſcribed about a ſolid figure, when either the angles, or ſides, or planes of the circumſcribed figure touch all the angles of the figure which it contains.

PROP.

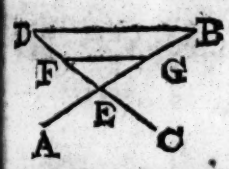
PROP. I.



One part AC of a right line cannot be in a plane ſuperficies, and another part CB elevated upward.

Produce AC in the plane directly to F. If you conceive CB to be drawn ſtrait from AC, then two right lines AB, AF, have one common ſegment AC. *a Which is impoſſible.* a 10. *ax.* 1.

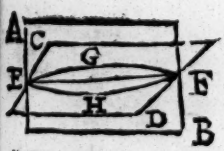
PROP. II.



If two right lines AB, CD, cut one another, they are in the ſame plane: And every triangle DEB is in one and the ſame plane.

For imagine EFG, part of the triangle DEB, to be in one plane, and the part FDGB to be in another, then EF part of the right line ED is in a plane, and the other part elevated upwards. *a Which is abſurd.* Therefore the triangle EDB a 1. 11. is in one and the ſame plane; and ſo alſo are the right lines ED, EB; *a* wherefore the whole lines AB, DC, are in one plane. *Which was to be demonſtrated.*

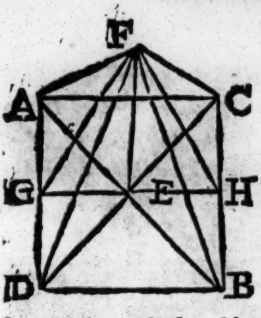
PROP. III.



If two planes AB, CD, cut one the other, their common ſection EF is a right line.

If EF the common ſection be not a right line, *a* then in the plane AB draw the right line EGF, *a* and a 1. *poſt.* 1. in the plane CD draw the right line EHF. therefore two right lines EGF, EHF, include a ſuperficies. *b Which is abſurd.* b 14. *ax.* 1.

PROP.



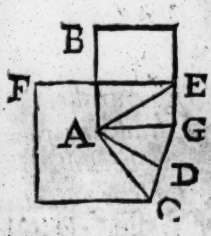
If a right line EF be at right angles erected upon two lines AB, CD, cutting one the other at the common ſection E; it ſhall alſo be at right angles to the plane ACBD drawn by the ſaid lines.

Take EA, EC, EB, ED, equal one to the other, and join the right lines AC, CB, BD, AD, draw any right line GH thro' E, and join FA, FC, FD, FB, FG, FH. Becauſe AE is *a* = EB, and DE *a* = EC, and the angle AED *b* = CEB. *c* therefore AD is = CB, *c* and likewiſe AC = DB. *d* therefore AD is parallel to CB, *d* and AC to DB. *e* wherefore the angle GAE = EBH, and the angle AGE = EHB. But alſo AE *f* = EB. *g* therefore GE = EH, *g* and AG = BH. whence by reaſon of the right angles, by the hyp. and ſo equal, at E, *h* the baſes FA, FC, FB, FD, are equal. Therefore the triangles ADF, FBC, are equilateral one to another. *k* and thence the angle DAF = BCF. Therefore in the triangles AGF, FBH, the ſides FG, FH *l* are equal; and ſo by conſequence the triangle FEG and FEH are mutually equilateral. *m* therefore the angles FEG, FEH are equal, and *n* ſo right angles. In like manner, FE makes right angles with all the lines drawn thro' E in the plane ADBC, *o* and is therefore perpendicular to the ſaid plane.

a conſtr.
b 15. 1.
c 4. 1.
d *ſch.* 34. 1.
e 29. 1.
f *conſtr.*
g 26. 1.
h 4. 1.
k 8. 1.
l 4. 1.
m 8. 1.
n 10. *def.* 1.
o 3. *def.* 11.

PROP.

PROP. V.

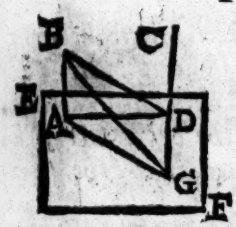


If a right line AB be erected perpendicular to three right lines AC, AD, touching one the other at the common ſection, thoſe three lines are in the ſame plane.

For AC, AD, *a* are in one plane FC; *a* and AD, AE, are in one plane BE. which if you conceive to be ſeveral planes, then let their interſection *b* be the right line AG; therefore becauſe BA by the Hypoth. is perpendicular to the right lines AC, AD. *c* and ſo to the plane FC, *d* it is alſo perpendicular to the right line AG. therefore (ſince *a* that AB is in the ſame plane with AC, AE) the angles BAG, BAE, are right angles, and conſequently equal, the part and the whole. *Which is abſurd.*

a 2. 11.
b 3. 11.
c 4. 11.
d 3. *def.* 11.

PROP. VI.

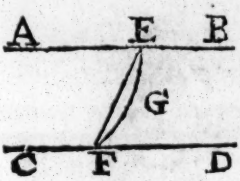


If two right lines AB, DC, be erected perpendicular to one and the ſame plane EF; thoſe right lines AB, DC are parallel one to the other.

Draw AD, whereunto let DG = AB be perpendicular in the plane EF, and join BD, BG, AG. Being in the triangles BAD, ADG, the angles BAD, ADG *a* are right angles, and AB *b* = DG, and AD is common, *c* therefore BD is = AG. whence in the triangles AGB, BGD, equilateral one to the other, the angle BAG is *d* = BDG; of which being BAG is a right angle, BDG ſhall be ſo alſo. but the angle GDC is ſuppoſed right, therefore the right line GD is perpendicular to the three lines DA, DB, CD. *e* which are therefore in the ſame plane *f* wherein AB is. Wherefore ſince AB and CD are in the ſame plane, and the internal angles BAD, CDA, are right angles, *g* AB and CD ſhall be parallels. *Which was to be dem.*

a *hyp.*
b *conſtr.*
c 4. 1.
d 8. 1.
e 5. 11.
f 2. 11.
g 28. 1.

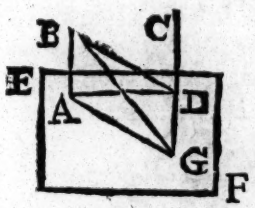
PROP. VII.



If there be two parallel right lines AB, CD, and any points E, F, be taken in both of them, the line EF which is joined at theſe points, is in the ſame plane with the parallels ABCD.

Let the plane in which AB, CD are, be cut by another plane at the points E, F. then if EF is not in the plane ABCD, it ſhall not be the common ſection. Therefore let EGF be the
a 3. 11. common ſection; which *a* then is a right line.
b 14. *ax.* 1. therefore two right lines EF, EGF, include a ſuperficies. *b Which is abſurd.*

PROP. VIII.

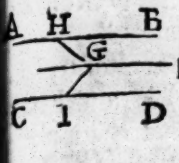


If there be two parallel right lines AB, CD, whereof one AB is perpendicular to a plane EF. then the other CD ſhall be perpendicular to the ſame plane EF.

The preparation and demonſtration of the ſixth of this Book being transfer'd hither; the angles GDA, and GDB
a 4. 11. are right angles: *a* therefore GD is perpendicu-
b 7. 11. lar to the plane, wherein are AD, DB (*b* in
c 3. *def.* 11. which alſo AB, CD are,) *c* therefore GD is perpendicular to CD. but the angle CDA is alſo
d 29. 1. *d* a right angle. *e* therefore CD is perpendicular
e 4. 11. to the plane EF. *Which was to be dem.*

PROP.

PROP. IX.



Right lines (AB, CD) which are parallel to the ſame right line EF, but not in the ſame plane with it, are alſo parallel one to the other.

In the plane of the parallels AB, EF, draw HG perpendicular to EF; alſo in the plane of the parallels EF, CD, draw IG perpendicular to EF. *a* therefore EG is perpendicular to the plane wherein HG, GI are; and AH, CI are perpendicular to the ſame plane. *c* therefore EH and CI are parallels. *Which was to be dem.*

a 4. 11.
b 8. 11.
c 6. 11.

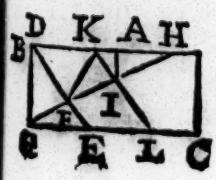
PROP. X.

If two right lines AB, AC, touching one another be parallel to two other right lines ED, DF, touching one another, and not being in the ſame plane, thoſe right lines contain equal angles, BAC, EDF.

Let AB, AC, DE, DF, be equal one to the other, and draw AB, BC, EF, BE, CF. Being AB, DE, *a* are parallels, and equal, *b* alſo BE, AD, are parallels and equal. In like manner CF, AD, are parallels and equal; *c* therefore alſo BE, FC, are parallels and equal. *d* Therefore BC, EF are equal. Wherefore ſince the triangles BAC, EDF, are of equal ſides one to the other, the angles BAC, EDF *e* ſhall be equal. *Which was to be dem.*

a *hyp. and conſtr.*
b 33. 1.
c 2. *ax.* 1. *and* 30. 1.
d 33. 1.
e 8. 1.

PROP. XI.



From a point given on high A, to draw a right line AI perpendicular to a plane below BC.

In the plane BC draw any

a 12. 1. line DE; to which from the point A *a* draw the
b 11. 1. perpendicular AF, and *b* likewiſe FH in the plane BC cutting the ſaid line DE at F; *a* then let fall AI perpendicular to FH. Which AI ſhall be perpendicular to the plane BC.

c 31. 1. For thro' I *c* let KIL be drawn parallel to DE.
d *conſtr.* Becauſe DE *d* is perpendicular to AF, and FH, e
e 4. 11. therefore DE ſhall be perpendicular to the plane
f 8. 21. IFA. and ſo alſo KL *f* is perpendicular to the
g 3. *def.* 11. ſame plane. *g* therefore the angle KIA is a right
h *conſtr.* angle. but the angle AIF is alſo *h* a right angle.
l 4. 11. *l* therefore AI is perpendicular to the plane BC. *Which was to be done.*

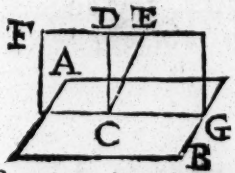
PROP. XII.

In a plane given BC, at a point given therein A, to erect a perpendicular line AF.

From ſome point without the
a 11. 11. plane D, *a* draw DE perpendicular to the ſaid plane BC, and joining the
b 31. 1. points A, E, by a line AE, *b* draw AF parallel
c 8. 11. to DE. *c* it is apparent that AF is perpendicular to the plane BC. *Which was to be done.*

This and the preceding Problem are practically performed by applying two ſquares to the point given; as appears by 4. 11.

PROP. XIII.

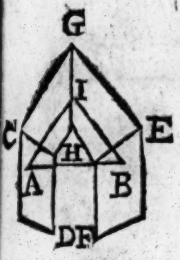


As a point given C in a plane given AB, two right lines CD, CE, cannot be erected perpendicular on the ſame ſide.

a 6. 11. For both CD, and CE, *a* ſhould then be perpendicular to the plane AB, and conſequently parallels; which is repugnant to the definition of parallel lines.

PROP.

PROP. XIV.

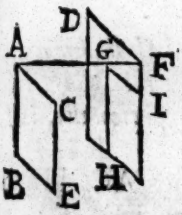


Planes CD, FE, to which the ſame right line AB is perpendicular, are parallel.

If you deny this; then let the planes CD, FE, meet, ſo that their common ſection be the right line GH, in which take any point I, draw to it the right lines IA, IB, in the ſaid planes, whereby in the triangle IAB, two angles IAB, IBA *a* are right angles. *b Which is abſurd.*

a *hyp.* and 3. *def.* 11.
b 17. 1.

PROP. XV.

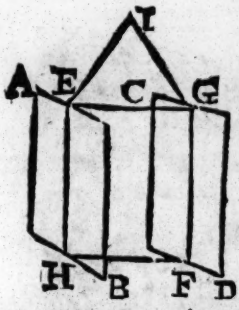


If two right lines AB, AC, touching one the other, be parallel to two other right lines DE, DF, touching one the other, and not being in the ſame plane with them, the planes BAC, EDF, drawn by thoſe right lines are parallel one to the other.

From A *a* draw AG perpendicular to the plane EF. *b* and let GH, GI be parallel to DE, DF. *c* theſe alſo ſhall be parallel to AB, AC. Therefore ſince the angles IGA, HGA, *d* are right angles, alſo CAG, BAG, *e* ſhall be right angles. *f* therefore GA is perpendicular to the plane BC; but the ſame is perpendicular to the plane EF. *h* therefore the planes BC, EF, are parallel. *Which was to be dem.*

a 11. 11.
b 31. 1.
c 30. 1.
d 3. *def.* 11.
e 29. 1.
f 4. 11.
g *conſtr.*
h 14. 11.

PROP. XVI.

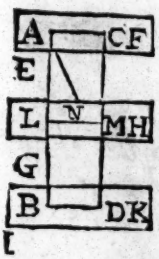


If two parallel planes AB, CD, be cut by ſome other plane HEIGF, their common ſections EH, GF are parallel one to the other.

For if they be conceived to be otherwiſe; being in the ſame plane that cuts them, they will meet ſomewhere, if produced; ſuppoſe in I; wherefore ſince the whole lines HEI, FGI *a* are in the planes AD, CD, being produced, the planes alſo ſhall meet. *contrary to the Hyp.*

a 1. 11.

PROP. XVII.



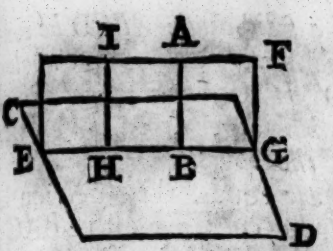
If two right lines ALB, CMD, be cut by parallel planes EF, GH, IK; they ſhall be cut proportionally, (AL. LB :: CM. MD.

Let the right lines AC, BD, be drawn in the planes EF, IK; as alſo AD paſſing thro' the plane GH in the point N. and join NL, LM the planes of the triangles ADC, ADB, make the ſections DB, LN, and AC, NM *a* parallels. Therefore AL. LB :: AN. ND *b* :: CM. MD. *Which was to be dem.*

a 16. 11.
b 2. 6.

PROP.

PROP. XVIII.

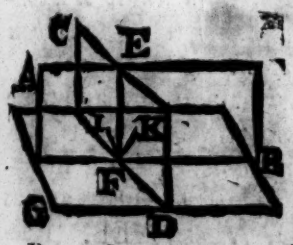


If a right line AB be perpendicular to ſome plane CD, all the planes extended by that right line AB (EF,&c.) ſhall be perpendicular to the ſame plane CD.

Let there be ſome plane EF drawn by AB, making the ſection EG with the plane CD; from ſome point whereof H, *a* draw HI parallel to AB in the plane EF; *b* then ſhall HI be perpendicular to the plane CD, and ſo likewiſe any other lines, that are perpendicular to EG. *b* therefore the plane EF is perpendicular to the plane CD; and by the ſame reaſon any other planes drawn by AB ſhall be perpendicular to EF. *Which was to be dem.*

a 31. 1.
b 8. 11.
c 4. *def.* 11.

PROP. XIX.

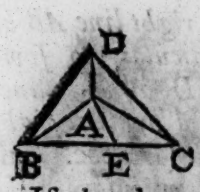


If two planes AB, CD, cutting one the other, be perpendicular to ſome plane GH. their line of common ſection EF ſhall be perpendicular to the ſame plane (GH.)

Becauſe the planes AB, CD, are taken perpendicular to the plane GH, it appears by 4. def. 11. *a* that out of the point F there may be drawn in both planes AB, CD, a perpendicular to the plane GH. which ſhall be *a* but one; and therefore the common ſection of the ſaid planes. *Which was to be dem.*

a 13. 11.

PROP. XX.

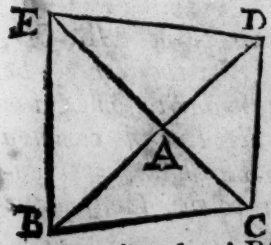


If a ſolid angle ABCD be contained under three plane angles, BAD, DAC, BAC, any two of them howſoever taken are greater than the third.

If the three angles are equal, the aſſertion is evident; if unequal, then let the greateſt be BAC; from whence *a* take away BAE = BAD, and
a 23. 1. make AD = AE; and alſo draw BEC, BD, DC.

Becauſe the ſide BA is common, and AD *b* = AE; and the angle BAE *b* = BAD. *c* thence is
b *conſtr.* BE = BD. but BD + DC is *d* ⊏ BC. *e* therefore
c 4. 1. DC ⊏ EC. Wherefore ſince AD *b* = AE, and the
d 20. 1. ſide AC is common, and DC ⊏ EC. *f* the angle
e *ax.* 1. CAD ſhall be ⊏ EAC. *g* therefore the angle
f 25. 1. BAD + CAD ⊏ BAC. *Which was to be dem.*
g 4. *ax.* 1.

PROP. XXI.



Every ſolid angle A is contained under leſs angles than four plane right angles.

For let a plane any-wiſe cutting the ſides of the ſolid angle A make a many-ſided figure BCDE, and as many triangles ABC, ACD, ADE, AEB. I denote all the angles of the polygone by X; and I term the ſum of the angle at the baſes of the
a 32. 1. & triangles Y. wherefore X + 4 right angles *a* =
ſch. 31. 1. Y + A. but being that, (of the angles at B) *b*
b 20. 11. the angle ABE + ABC is ⊏ CBE, and the ſame
c 5. *ax.* 1. is true alſo of the angles at C, at D, and at E, *c* it is manifeſt that Y is ⊏ X. and conſequently A ſhall be ⊐ 4 right angles. *Which was to be dem.*

PROP.

PROP. XXII.

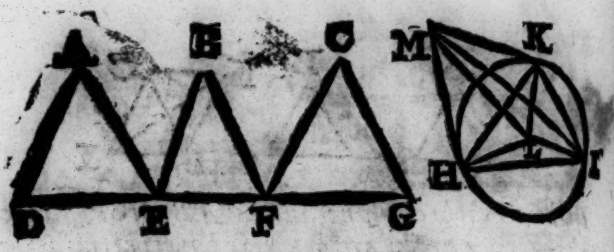
If there be three plane angles A, B, HCI, whereof two howſoever taken are greater than the third, and the right lines which contain them be equal AD, AE, FB, &c. then of the right lines DE, FG, HI, coupling thoſe equal right lines together, it is poſſible to make a triangle.

A triangle may be *a* made of them, if any two be greater than the third: but they are ſo. For *b* make the angle HCK = B, and CK = CH, and draw HK, IK *c* thence HK = FG. and becauſe the angle KCI *d* ⊏ A. *e* therefore KI ⊏ DE. but KI *f* ⊐ HI + KH (FG.) therefore DE ⊐ HI + FG. By the like argument any two may be proved greater than the third; and conſequently *a* it is poſſible to make a triangle of them. *Which was to be dem.*

a 22. 1.
b 23. 1.
c 4. 1.
d *hyp.*
e 24. 1.
f 20. 1.

PROP.

PROP. XXIII.



To make a ſolid angle MHIK of three plane angles A, B, C, whereof two howſoever taken are greater than the third. * *But it is neceſſary that thoſe three angles be leſs than four right angles.*

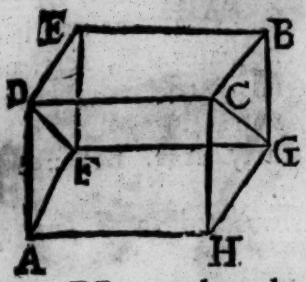
* 21. 11.

Make AD, AE, BE, BF, CF, CG, equal one to the other; and of the ſubtended lines DE, EF, FG (that is, of the equal lines HI, IK, KH) a make the triangle HKI; about which *b* deſcribe the circle LHKI. * But becauſe AD is ⊏ HL. *c* let ADq be = HLq + LMq. *d* and let LM be perpendicular to the plane of the circle HKI. and draw HM, KM, IM. wherefore ſince the angle HLM *e* is a right angle, *f* thence is MHq = HLq + LMq *g* = ADq. therefore MH = AD. By the ſame reaſon MK, MI, AD (that is, AE, EB, &c.) are equal; therefore ſince HM = AD, and MI = AE, and DE = HI. *k* the angle A ſhall be = HMI, *k* as likewiſe the angle IMK = B, *k* and the angle HMK = C. wherefore a ſolid angle is made at M of the three given plane angles. *Which was to be done.* AD is aſſumed to be ⊏ HL. But this is manifeſt. For if AD be = or ⊐ HL, then is the angle A *l* = *m* or ⊏ HLI. In like manner ſhall B be = or ⊏ HLK, and C = or ⊏ KLI. wherefore A+B+C * ſhall either equal or exceed four right angles. *contrary to the Hypoth.* therefore rather let AD be ⊏ HL. *Which was to be dem.*

a 22. 11. and 22. 1.
b 5. 4.
* *See Clavius.*
c *ſch.* 47. 1.
d 12. 11.
e 3. *def.* 11.
f 47. 1.
g *conſtr.*
h *conſtr.*
k 8. 1.
l *conſtr.* & 8. 1.
m 22. 1.
* 4. *cor.* 13. 1.

PROP.

PROP. XXIV.

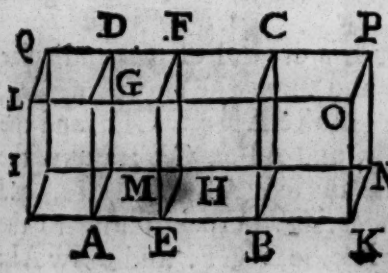


If a ſolid AB be contained under parallel planes, the oppoſite planes thereof (AG, DB, &c.) are like and equal parallelograms.

The plane AC cutting the parallel planes AG, DB, *a* makes the ſections AH, DC, parallels. and by the ſame reaſon AD, HC are parallels. Therefore ADCH is a parallelogram. By the like argument the other planes of the parallelepipedon are *b* parallelograms. wherefore being AF is parallel to HG, and AD to HC, *c* the angle FAD ſhall be = CGH. therefore becauſe AF *d* = HG, and AD *d* = HC, and ſo AF. AD :: HG. HC, the triangles FAD, GHC, *g* are like and *h* equal; and conſequently the parallelograms AE, HB are like and *k* equal. and the ſame may be ſhewn of the other oppoſite planes. therefore, *&c.*

a 16. 11.
b 35. *def.* 1.
c 10. 11.
d 34. 2.
e 7. 5.
g 6. 6.
h 4. 1.
k 6. *ax.* 1.

PROP. XXV.



If a ſolid Parallelepipedon ABCD be cut by a plane EF, parallel to the oppoſite planes AD, BC; then as the baſe AH is to the baſe BH, ſo ſhall ſolid AHD be to ſolid BHC.

Conceive the Parallelepipedon to be extended on either ſide, and take AI = AE, and BK = EB, and put the plane IQ, KP, parallel to the planes AD, BC; then the pgrs. IM, AH, and *a* DL, DG, *b* and IQ, AD, EF, *&c.* are *a* like and equal. *c* wherefore the Parallelepipedon AQ is = AF; and by

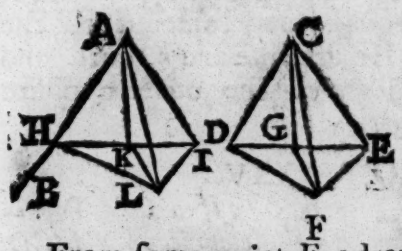
a 36. 1. *&* 1. *def.* 6.
b 24. 11.
c 10. *def.* 11.

by the ſame reaſon the Parallelepipedon BP = BF. therefore the ſolids IF, EP are as multiple of the ſolids AF, EC, as the baſes IH, KH, are of the baſes AH, BH. And if the baſis IH be
d 24. 11. □, =, □ KH, *d* likewiſe ſhall the ſolid IE be
and 9. *def.* □, =, □ EP. *e* conſequently AH. BH :: AF.
11. EC. *Which was to be dem.*
e 6. *def.* 5. *The ſame may be accommodated to all ſorts of Priſms, whence*

Coroll.

If any Priſm whatſoever be cut by a plane parallel to the oppoſite planes, the ſection ſhall be a figure equal and like to the oppoſite planes.

PROP. XXVI.

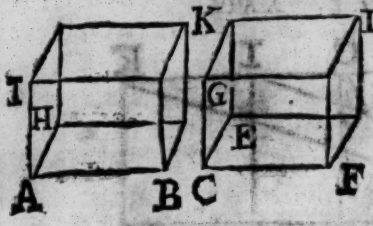


Upon a right line given AB, and at a point given in it A, to make a ſolid angle AHIL equal to a ſolid angle given CDEF.

a 11. 11. From ſome point F *a* draw FG perpendicular to the plane DCE, and draw the right lines DF, FE, EG, GD, CG. Make AH = CD, and the angles HAI = DCF, and AI = CE; and in the plane HAI make the angle HAK = DCG, and AK = CG. then erect KL perpendicular to the plane HAI, and let KL be = GF. and draw AL: then AHIL ſhall be a ſolid angle equal to that given CDEF. For the conſtruction of this does wholly reſemble the framing of that, as may eaſily appear to any that examine it.

PROP.

PROP. XXVII.



Upon a right line given AB to deſcribe a parallelepipedon AK, like, and in like manner ſituate, with a ſolid parallelepipedon given CD.

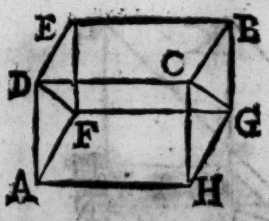
Of the plane angles BAH, HAI, BAI, which are equal to FCE, ECG, FCG, *a* make the ſolid angle A equal to the ſolid angle C. alſo *b* make FC. CE :: BA. AH. *b* and CE. CG :: AH. AI (*c* whence of equality FC.CG :: BA. AI) and finiſh the parallelepipedon AK, which ſhall be like to that which is given.

a 26. 11.
b 12. 6.
c 22. 5.

For by the conſtruction, the Parallelogram *d* BH is like FE, and *d* HI to EG, and *d* BI to FG, and *e* ſo the oppoſites of theſe to the oppoſites of them: therefore the ſix planes of the ſolid AK are like to the ſix planes of the ſolid CD, *f* and conſequently AK, CD, are like ſolids. *Which was to be dem.*

d 1. *def.* 6.
e 14. 11.
f 9. *def.* 11.

PROP. XXVIII.



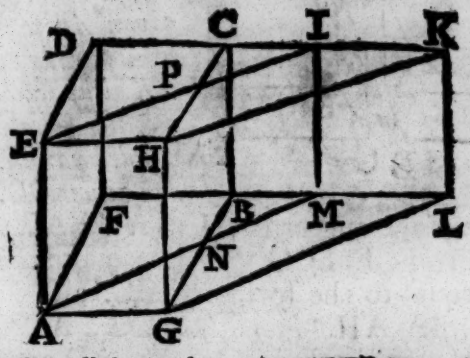
If a ſolid parallelepipedon AB be cut by a plane FGCD drawn by the diagonal lines DF, CG, of the oppoſite planes AE, HB, that ſolid AB ſhall be equally biſected by the plane FGCD.

For becauſe DC, FG, are *a* equal and parallels, *b* the plane FGCD is a Pgr. and being *a* the Pgrs. AE, HB, are equal and like, *b* alſo the triangles AFD, HGC, CGB, DFE are equal and like. But the Pgrs. AC, AG, are equal and like to FB and FD. therefore all the planes of the priſme FGCDAH are equal and like to all the planes of the priſme FGCDEB, and *c* conſequently this priſme is equal to that. *Which was to be dem.*

a 24. 11.
b 34. 1.
c 9. *def.* 11.

PROP.

PROP. XXIX.



*Solid Parallelepipedons AGHEFBCD, AGHEMLKI, being conſtituted upon the ſame baſe AGHE, and * in the ſame height, whoſe inſiſting lines AF, AM, are placed in the ſame right lines AG, FL, are equal one to the other.*

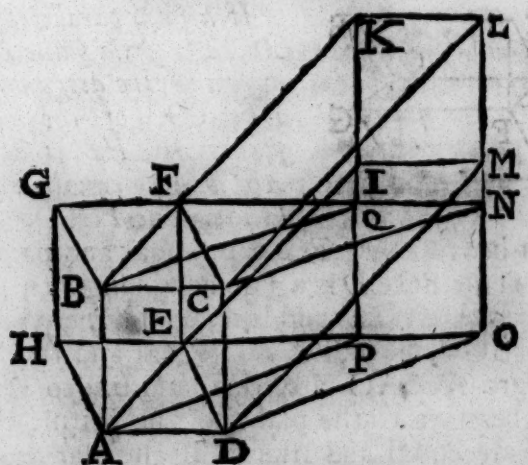
* *i. e. between the parallel planes AGHE, FLKD, and ſo underſtand it in the following.*

For *a* if from the equal priſmes AFMEDI, GBLHCK, the common priſme NBMPCI be taken away, and the ſolid AGNEHR be added, the Parallelepipedon AGHEFBCD ſhall be = it in the AGHEMLKI. *Which was to be dem.*

a 10. *def.* 11. *and* 35. 1.

b 3. *and* 1. *ax.* 1.

PROP. XXX.



Solid parallelepipedons ADBCHEFG, ADCBIMLK being conſtituted upon the ſame baſe ADBC, and

in

in the same height, whose insisting lines AH, AI are not placed in the same right lines, are equal one to the other.

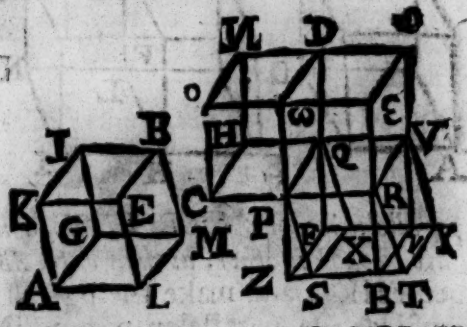
For produce the right lines HEO, GFN, and LMO, KIP; and draw AP, DO, BQ, CN. *a* then shall DC, AB, HG, EF, PQ, ON be as well equal and parallel one to the other as AD, HE, GF, BC, KL, IM, QN, PO. *b* wherefore the parallelepipedon ADCBPONQ shall be equal to either parallelepipedon ADCBHEFG, ADCBIMLK; and *c* consequently these two are equal one to the other. *Which was to be dem.*

a 34. 1.

b 29. 11.

c 1. *ax.* 1.

PROP. XXXI.



Solid parallelepipedons, ALEKGMBI, CPωOHQDN, being constituted upon equal bases ALEK, CPωO, and in the same height are equal, one to the other.*

First, let the parallelepipedons AB, CD, have the sides perpendicular to the bases, and at the side CP being produced, *a* make the parallelogram PRTS equal and like to the parallelogram KELA. *b* and so the parallelepipedon PRTSQVYX equal and like to the parallelepipedon AB. Produce OωE, NDϛ, ωPZDQF, ERB, ϛVγ, TSZ, YXF, and draw Eϛ, Bγ, ZF.

The planes OEϛN, CRVHZTYF *c* are parallels one to the other; *d* and the pgrs. ALEK, CDωO, PRTS, PRBZ are equal. Therefore since the parallelepipedon CD.PVϛω :: pgr. Cω(PRBZ) P*e* :: parallelepipedon PRBZQVγF. PVϛω; the paral-

* *by height understand the perpendicular drawn from the plane of the base to the opposite plane.*

a 18. 6.

b 27. 11.

and 10. *def.* 11.

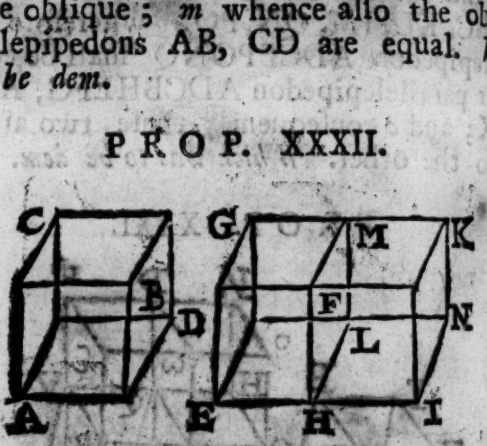
c 30. *def.* 11.

d *hyp. and* 35. 1.

e 25. 11.

f 9. 5. parallelepipedon CDf ſhall be = PRBZVQ, γ Fg =
g 29. 11. PRVQSTYX *h* = AB. *Which was to be dem.*
h *conſtr.* But if the parallelepipedons AB, CD, hav
ſides oblique to the baſes, then on the ſame ba
ſes and in the ſame heighth place parallelepip
k 29. 11. dons whoſe ſides are perpendicular to the baſe
k They ſhall be equal to one another, and thoſ
m 1. *ax.* 1. that are oblique; *m* whence alſo the oblique
parallelepipedons AB, CD are equal. *Which was to be dem.*

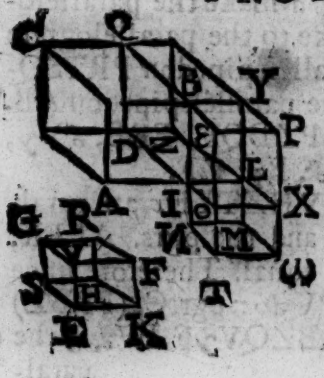
PROP. XXXII.



Solid parallelepipedons ABCD, EFGL, of the ſame heighth, are one to the other, as their baſes AB, EF.

a 45. 1. Produce EHI, *a* and make the pgr. FI = AB
b 31. 1. and *b* compleat the parallelepipedon FINM. It
c 31. 11. is clear that the parallelepipedon FINM. (
d 25. 11. ABCD.) EFGL. *d* :: FI (AB.) EF. *Which was to be dem.*

PROP. XXXIII.



Like ſolid parallelepipedons, ABCD, EFGH, are in tripled proportion one to the other of that in which their homologous ſides or like proportion AI, EK are.

Produce the right lines AIL, DIO, BIN, and make IL, IO, IN, equa
a 3. 1.

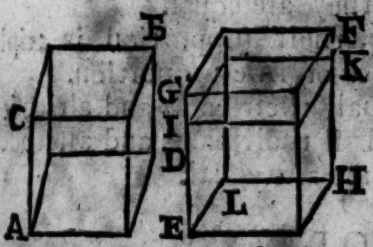
to EK, KH, KF, *b* and ſo the parallelepipedon IXMT equal and like to the parallelepipedon EFGH. *c* Let the paralleps. IXPB, DLYQ be finiſhed. *d* Then ſhall be AL. IL (EK) :: DI. IO (HK) :: BI. IN. KF. *e* that is, the Pgr. AD. DL :: DL. IX :: BO. IT. *f i. e.* the parallepp. ABCD. DLQY :: DLQY. IXBP :: IXBP. IXMT. (*g* EFGH.) *b* therefore the proportion of ABCD to EFGH is triple of the proportion of ABCD to DLQY, *k* or of AI to EK. *Which was to be dem.*

b 27. 11. c 31. 1. d *hyp.* e 1. 6. f 32. 11. g *conſtr.* h 10. *def.* 5. k 1. 6.

Coroll.

Hence it appears, that if four right lines be continually proportional, as the firſt is to the fourth, ſo is a parallelepipedon deſcribed on the firſt to a parallelepipedon deſcribed on the ſecond, being like and in like manner deſcribed.

PROP. XXXIV.



In equal ſolid parallelepipedons ADCB, EHGF, the baſes and altitudes are reciprocal (AD. EH :: EG. AC.) And ſolid parallelepipedons, ADCB, EHGF, whoſe baſes and altitudes are reciprocal, are equal.

Firſt, let the ſides CB, GE be perpendicular to the baſes; then if the altitudes of the ſolids are equal, the baſes alſo ſhall be equal, and the thing is clear. But if the altitudes are unequal, from the greater EG *a* take EI = AC, and at I *b* draw the plane IK parallel to the baſe EH. then

a 3. 1. b 31. 1. c 32. 11. d 17. 5. e 1. 6. f *conſtr.*

1. *Hyp.* AD. EH *c* :: parallepp. ADCB. EHIK *d* :: parallepp. EHGF. EHIK *c* :: GL. IL *e* :: GE. IE. (*f* AC.) *g* it is plain therefore that AD. EH :: GE. AC. *Which was to be dem.*

g 11. 5. h 32. 11. k *hyp.* l 1. 6.

2. *Hyp.* ADCB. EHIK *b* :: AD. EH *k* :: EG. EI *l* :: GL. IL *m* :: parallepp. EHGF. EHIK. *n* where-

m 32. 11. n 9. 5.

fore the parallelepipedon ADCB = EHGF. *Which was to be dem.*

Moreover, let the sides be oblique to the bases, and erect right parallelepipedons upon the same bases in the same altitude; the oblique parallelepipedons shall be equal to them. Wherefore since by the first part, the bases and altitudes of those be reciprocal, the bases and altitudes of these also shall be reciprocal. *Which was to be dem.*

Coroll.

All that hath been demonstrated of parallelepipedons in the 29,30,31,32,33,34 *Prop. does also agree in triangular prisms, which are half parallelepipedons, as appears by Prop.* 28. Therefore,

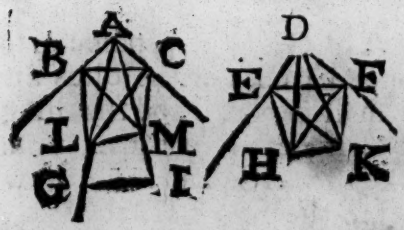
1. Triangular prisms are of equal heighth with their bases.

2. If they have the same or equal bases and the same altitude, they are equal.

3. If they be like, their proportion is triple to that of their sides of like proportion.

4. If they be equal, their bases and altitudes are reciprocal; and if their bases and altitudes be reciprocal, they are also equal.

PROP. XXXV.



If there be two plane angles BAC, EDF, equal, and from the points of those angles two right lines AG, DH be elevated on high, containing equal angles with the lines first given, each to his correspondent angle (the angle GAB = HDE, and GAC = HDF.) and if in those elevated lines AG, DH, some points be taken, G, H; and from these points perpendicular lines GI, HK, drawn to the planes BAC, EDF, in which the angles first given are, and right lines

lines AI, DK, be drawn to the angles first given from the points I, K, which vre made by the perpendiculars in the planes; those right lines with the elevated lines AG, DH shall contain equal angles GAM, HDK.

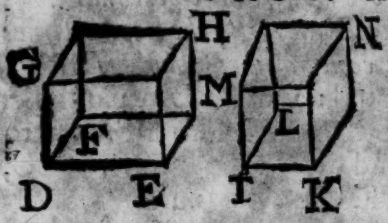
Make DH, AL, equal; and GI, LM parallels, and MC to AC, MB to AB, KF to DF, KE to DE perpendicular; and draw the right lines BC, LB, LC, and EF, HF, HE; *a* and LM is perpendicular to the plane BAC; *b* wherefore the angles LMC, LMA, LMB; and by the same reason the angles HKF, HKD, HKE are right angles. Therefore ALq *c* = LMq + AMq *c* = LMq+CMq + ACq *c* = LCq + ACq. *d* therefore the angle ACL is a right angle. Again ALq *e* = LMq + MAq *e* = LMq + BMq + BAq *e* = BLq + BAq. *d* therefore the angle ABL is also a right angle. By the like inference the angles DFH. DEH are right angles; *f* therefore AB=DE, *f* and BL=EH, *f* and AC =DF, and CL=FH. *g* wherefore also BC = EF; *g* and the angle ABC = DEF, *g* and the angle ACB = DFE. whence the other right angles CBM, BCM, are equal to the other FEK, EFK. *k* therefore CM = FK, *l* and so also AM = DK. therefore if from LAq *m* = HDq be taken away AMq = DKq, *n* there remains LMq = HKq. wherefore the triangles LAM, HDK are equilateral one to the other; *o* therefore the angle LAM = HDK. *Which was to be dem.*

a 8. 11.
b 3. *def.* 11.
c 47. 1.
d 46. 1.
e 47. 1.
f 26. 1
g 4. 1.
h 1 *ax.* 1.
k 26. 1.
l 47. 1.
m *constr.*
n 47. 1. & 3. *ax.*
o 8. 1.

Coroll.

Therefore, if there be two plane angles equal, from whose points equal right lines be elevated on high, containing equal angles with the lines first given, each to each; perpendiculars drawn from the extreme points of those elevated lines to the planes of the angles first given, are equal one to the other; *viz.* LM = HK.

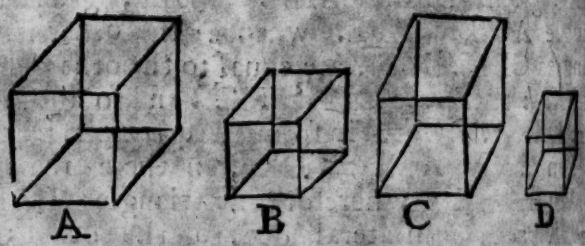
PROP. XXXVI.



If there be three right lines DE, DG, DF proportional, the ſolid parallelepipedon DH made of them, is equal to the ſolid parallelepipedon IN made of the middle DG(IL) which is alſo equilateral, and equiangular to the ſaid parallelepipedon DH.

a *hyp.* Becauſe DE. IK *a*:: IL. DF. *b* the parallelo-
b 14. 6. gram LK ſhall be = FE. and by reaſon of the equality of the plane angles at E and I, and of the lines GD, IM, alſo the altitudes of the pa-
c 31. 11. rallelepipedons are equal by the preceding Corollary, therefore the parallelepipedons are equal one to the other. *Which was to be dem.*

PROP. XXXVII.



If there be four right lines A,B,C,D, proportional, the ſolid parallelepipedons A,B,C,D being like, and in like ſort deſcribed from them ſhall be proportional. And if the ſolid parallelepipedons, being like and in like ſort deſcribed, be proportional (A. B :: C. D) then thoſe right lines A,B,C,D, ſhall be proportional.

For the proportions of the parallelepipedons
a 33. 11. *a* are triple of thoſe of the lines; therefore if A.
b *ſch.* 23. 5. B :: C. D. *b* then ſhall the parallelepipedon A. parallelepipedon B :: parallelepipedon C. parallelepipedon D. and ſo alſo contrarily.

PROP.

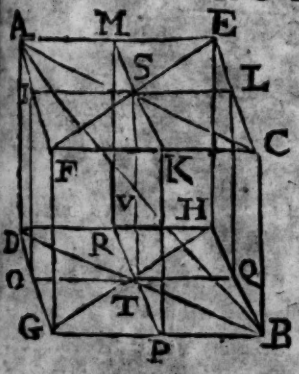
PROP. XXXVIII.

If a plane AB be perpendicular to a plane AC, and a perpendicular line EF be drawn from a point E in one of the planes (AB) to the other plane AC, that perpendicular EF ſhall fall upon the commen ſection of the planes AD.

If it be poſſible, let F fall without the interſection AD, and in the plane AC *a* draw FG perpendicular to AD, and join EG. The angle FGE *b* is a right angle, and EFG is ſuppoſed to be ſuch alſo; therefore two right angles are in the triangle EFG. *c Which is abſurd.*

a 12. 1.
b 4. *and* 3. *def.* 11.
c 17. 1.

PROP. XXXIX.



If the ſides (AE, FC, AF, EC, and DH, GB, DG, HB) of the oppoſite planes AC, DB, of a ſolid parallelepipedon AB, be divided into two equal parts and planes IL, QO, PKMR, be drawn thro' their ſections, the common ſection of the planes ST, and the diameter of the ſolid parallelepipedon AB ſhall divide one the other into two equal parts.

Draw the right lines SA, SC, TD, TB. Becauſe *a* the ſides DO, OT are equal to the ſides BQ, QT. *b* and the alternate angles TOD, TQB equal alſo, *c* the baſe *e* DT, TB, and the angles DTO, BTQ are equal. *d* therefore DTB is a right line, and ſo in like manner is ASC. Moreover *e* as well AD is parallel and equal to FG *e* as FG to CB, and *f* thence AD is parallel and equal to C;

a 34. 1.
b 29. 1.
c 4. 1.
d *ſch.* 15. 1.
e 34. 1.
f 9. 11. &
1. *ax.*
g 33. 1.
h 7. 11.

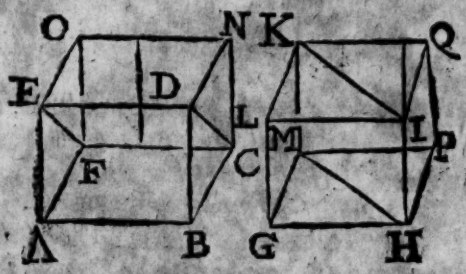
g 33. 1. h 7. 11. k 7. *ax.* 1. l 26. 1.

g and consequently AC to DB. *h* wherefore AB and ST are in the same plane ABCD. Therefore since the vertical angles AVS, BVT, and the alternate angles ASV, BTV are equal; *k* and AS = BT; therefore shall AV be = BV. *l* and SV = VT. *Which was to be dem.*

Coroll.

Hence in every parallelepipedon all the diameters bisect one another in one point, V.

PROP. XL.



If two Prisms ABCFED, GHMLIK, be of equal altitude, whereof one hath its base ABCF a parallelogram, and the other GHM, a triangle; and if the parallelogram ABCF be double to the triangle GHM; these Prisms ABCFED, GHMLIK are equal.

a 31. 11. b 34. 1. & 7. *ax.* c *hyp.* d 28. 11. e 7. *ax.* 1. *Andr. Tacq.*

For if the parallelepipedons AN, GQ be completed, *a* they shall be equal, because of the equality *b* of the bases AC, GP, and *c* of the altitudes. *d* therefore also the prisms, *e* the halfs thereof shall be equal. *Which was to be dem.*

Coroll.

From the preceding demonstrations, the dimension of triangular Prisms, and quadrangular, or parallelepipedons, is learnt; viz. *by multiplying the altitude into the base.*

As if the altitude be 10 feet, and the base 100 square feet (the base may be measured by *sch.* 35. 1. or by 41. 1.) then multiply 100 by 10, and

1000

1000 cubic feet ſhall be produced, for the ſolidity of the priſm given.

For as a rectangle, ſo alſo is a right parallelepipedon produced of the altitude multiplied into the baſe. Therefore every parallelepipedon is produced of the altitude multiplied into the baſe, as appears by 31. of this Book.

Moreover, ſince the whole parallelepipedon is produced of the altitude drawn into the baſe, the half thereof (that is, a triangular Priſm) ſhall be produced of the altitude drawn into half the baſe, namely the triangle.

An Advertiſement.

Obſerve, That of thoſe letters which denote a ſolid angle, the firſt is always at the point in which the angle is; but of thoſe letters which denote a Pyramide, the laſt is at the ſupreme point thereof.

Ex. gr. The ſolid angle ABCD is at the point A; and the ſupreme point of the Pyramide BCDA is at the point A. and the baſe is the triangle BCD.

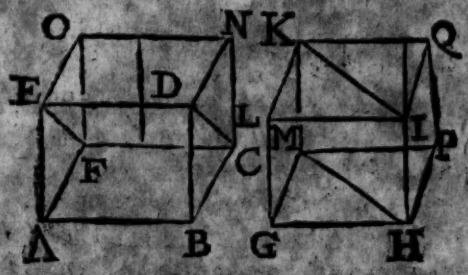
The End of the eleventh Book.

g 33. 1. *g* and consequently AC to DB. *h* wherefore AB
h 7. 11. and ST are in the same plane ABCD. Therefore
since the vertical angles AVS, BVT, and the al-
k 7. *ax.* 1. ternate angles ASV, BTV are equal; *k* and AS
l 26. 1. = BT; therefore shall AV be = BV. *l* and SV
= VT. *Which was to be dem.*

Coroll.

Hence in every parallelepipedon all the diameters bisect one another in one point, V.

PROP. XL.



If two Prisms ABCFED, GHMLIK, be of equal altitude, whereof one hath its base ABCF a parallelogram, and the other GHM, a triangle; and if the parallelogram ABCF be double to the triangle GHM; these Prisms ABCFED, GHMLIK are equal.

a 31. 11. For if the parallelepipedons AN, GQ be com-
b 34. 1. & pleted, *a* they shall be equal, because of the equa-
7. *ax.* lity *b* of the bases AC, GP, and *c* of the altitudes.
c *hyp.* *d* therefore also the prisms, *e* the halfs thereof
d 28. 11. shall be equal. *Which was to be dem.*
e 7. *ax.* 1.
Andr. *Coroll.*
Tacq. *From the preceding demonstrations, the dimension of triangular Prisms, and quadrangular, or parallelepipedons, is learnt*; viz. *by multiplying the altitude into the base.*

As if the altitude be 10 feet, and the base 100 square feet (the base may be measured by *sch.* 35. 1. or by 41. 1.) then multiply 100 by 10, and

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1000 cubic feet ſhall be produced, for the ſolidity of the priſm given.

For as a rectangle, ſo alſo is a right parallelepipedon produced of the altitude multiplied into the baſe. Therefore every parallelepipedon is produced of the altitude multiplied into the baſe, as appears by 31. of this Book.

Moreover, ſince the whole parallelepipedon is produced of the altitude drawn into the baſe, the half thereof (that is, a triangular Priſm) ſhall be produced of the altitude drawn into half the baſe, namely the triangle.

An Advertiſement.

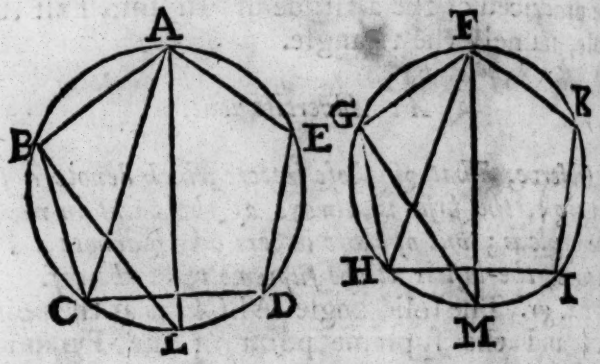
Obſerve, That of thoſe letters which denote a ſolid angle, the firſt is always at the point in which the angle is; but of thoſe letters which denote a Pyramide, the laſt is at the ſupreme point thereof.

Ex. gr. The ſolid angle ABCD is at the point A; and the ſupreme point of the Pyramide BCDA is at the point A. and the baſe is the triangle BCD.

The End of the eleventh Book.

THE TWELFTH BOOK OF EUCLIDE's ELEMENTS.

PROP. I.



Like polygonous figures ABCDE, FGHIK, described in circles ABD, FGI, are one to another, as the squares described of the diameters of the circles AL, FM.

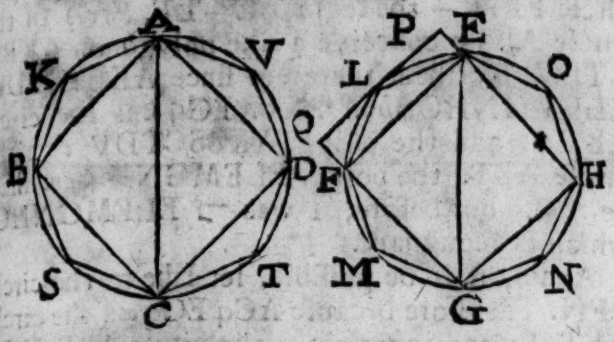
Draw AC, BL, FH, GM. Because
a 1. *def.* 6. *a* the angle ABC = FGH, *a* and AB. BC :: FG.
b 6. 6. GH. *b* therefore shall the angle ACB (*c* ALB)
c 21. 3. be = FHG (*c* FMG.) but the angles ABL,
d 31. 3. FGM *d* are right and so equal; *e* therefore the
e 32. 3. triangles ABL, FGM are equiangular. *f* where-
f *cor.* 4. 6. fore AB. FG :: AL. FM. *g* therefore ABCDE.
g 22. 6. FGHIK :: ALq. FMq.

Coroll.

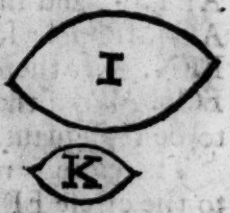
Hence (because AB. FG :: AL. FM :: BC. GH, *&c.*) the contents of like polygonous figures de-
h 1. 12. & 12. 5. scribed in a circle are in *h* proportion as the diameters.

PROP.

PROP. II.



Circles ABT, EFN, are in proportion one to another, as the ſquares of their diameters AC, EG are.



Suppoſe ACq. EGq :: the circle ABT. I. I ſay then I is equal to the circle EFN.

For firſt, if it be poſſible, let I be leſs than the circle EFN, and let K be the exceſs or difference. Inſcribe the ſquare EFGH in the circle EFN, *a* it being the half of a circumſcribed ſquare, and ſo greater than the ſemicircle. *b* Divide equally in two the arches EF, FG, GH, HE, and at the points of the diviſions join the right lines EL, LF, *&c.* at L draw the tangent PQ (*c* which is parallel to EF) and produce HEP, GFQ. then is the triangle ELF *d* the half of the parallelogram EPQF, and ſo greater than the half of the ſegment ELF; and in like ſort the reſt of thoſe triangles exceed the halfs of the reſt of the ſegments. And if the arches EL, LF, FM, *&c.* be again biſected, and the right lines joined, the triangles will likewiſe exceed the half of the ſegments. Wherefore if the ſquare EFGH be taken from the circle EFN, and the triangles from the other ſegments, and this be done continually, at length *e* there will remain ſome magnitude leſs than K. Let us have gone ſo far, namely to the ſegments EL, LF, FM, *&c.* taken

a *ſch.* 7. 4.
b 30. 3.
c *ſch.* 27. 3.
d 41. 1.
e 1. 10.

f *hyp. and* 3. *ax.* taken together leſs than K. Therefore I (*f* the circle FENK) ⊐ the polyg. ELFMNHO (the
g 30. 3. & 1. *poſt.* circle FEN — the ſegm. EL + LF, &c.) In the circle ABT *g* conceive a like polygonon AKBS-
h 1. 12. CTDV inſcribed. therefore ſince AKBSCTDV.
k *hyp.* ELFMGNHO *b* :: ACq. EGq *k* :: the circle
l 9. *ax.* 1. ABT. I. and the polyg. AKBSCTDV *l* ⊐ the
m 14. 5. circle ABT. the polyg. ELFMGNHO *m* ſhall be ⊐ I. but before, I was ⊐ ELFMGNHO, which is repugnant.

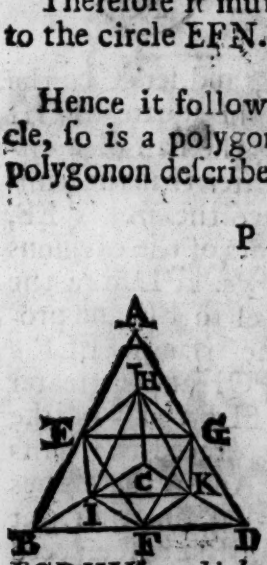
Again, if it be poſſible, let I be ⊏ the circle
n *hyp.* EFN. Therefore becauſe ACq. EGq *n*:: the circle ABT. I.; and inverſely I. the circle ABT. EGq. ACq. ſuppoſe I. the circle ABT :: the circle
o 14. 5. EFN. K. *o* therefore the circle ABT ⊏ K. *p* and
p 11. 5. EGq. ACq :: the circle EFN. K. which is ſhewn to be repugnant.

Therefore it muſt be concluded, that I is = to the circle EFN. *Which was to be dem.*

Coroll.

Hence it follows, that as a circle is to a circle, ſo is a polygonon deſcribed in one to a like polygonon deſcribed in the other.

PROP. III.



Every Pyramide ABDC having a triangular baſe, may be divided into two Pyramides AEGH, HIKC, equal, and like one to the other, having baſes triangular, and like to the whole ABDC; and into two equal Priſms, BFGEIH, EGDHIK; which two Priſms are greater than the half of the whole Pyramide ABDC.

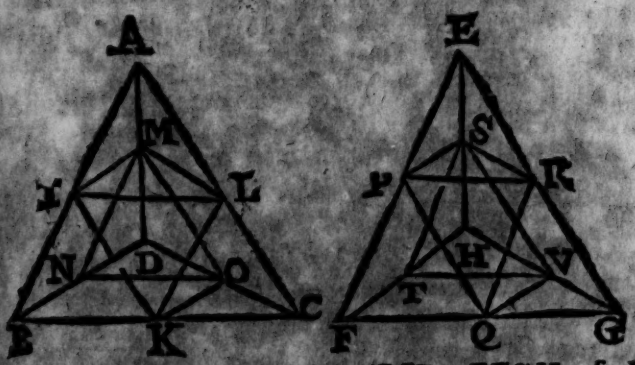
Divide the ſides of the pyramide into two parts at the points E, F, G, H, I, K, and join the right lines EF, FG, GE, EI, IF, FK, KG, GH, HE. Becauſe the ſides of the pyramide are proportionally

cut,

cut, *a* thence HI,AB; and GF, AB; and IF,DC; and HG, DC, *&c.* are parallels, and consequently HI,FG; and GH,FI are also parallels. therefore it is apparent that the triangles ABD, AEG, EBF, FDG, HIK, *b* are equiangular, and that the four last are *c* equal: in like manner the triangles ACB, AHE, EIB, HIC, FGK are equiangular; and the four last are equal one to the other. Also the triangles BFI, FDK, IKC, EGH; and lastly, the triangles AHG, GDK, HKC, EFI are like and equal. Moreover the triangles HIK to ADB, and EGH to BDC, and EFI to ADC, and FGK to ABC, *d* are parallel. From whence it evidently follows, first that the pyramides AEGH, HIKC are equal, and *e* like to the whole ABDC, and to one another. Next, that the solids BFGEIH, FGDIHK, are prisms, and that of equal heighth, as being placed between the parallel planes ABD, HIK, but the base BFGE is *f* double of the base FDG, wherefore the said prisms are equal; whereof the one BFGEIH is greater than the pyramide BFEI, that is, then AEGH, the whole than its part; and consequently the two prisms are greater than the two pyramides, and so exceed the half of the whole pyramide ABDC. *Which was to be dem.*

a 2. 6.
b 29. 1.
c 26. 1.
d 15. 11.
e 10. *def.* 11.
f 2. *ax.* 1.
g 40. 11.

PROP. IV.



If there be two pyramides ABCD, EFGH, of the same altitude, having triangular bases ABC, EFG; and

f *hyp. and* 3. *ax.* taken together leſs than K. Therefore I (*f* the circle FENK) ⊐ the polyg. ELFMNHO (the
g 30. 3. *&* 1. *poſt.* 1. circle FEN — the ſegm. EL + LF, *&c.*) In the circle ABT *g* conceive a like polygonon AKBS-
h 1. 12. CTDV inſcribed. therefore ſince AKBSCTDV.
k *hyp.* ELFMGNHO *h* :: ACq. EGq *k* :: the circle
l 9. *ax.* 1. ABT. I. and the polyg. AKBSCTDV *l* ⊐ the
m 14. 5. circle ABT. the polyg. ELFMGNHO *m* ſhall be ⊐ I. but before, I was ⊐ ELFMGNHO. which is repugnant.

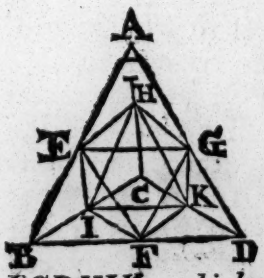
Again, if it be poſſible, let I be ⊏ the circle
n *hyp.* EFN. Therefore becauſe ACq. EGq *n* :: the circle ABT. I.; and inverſely I. the circle ABT. EGq. ACq. ſuppoſe I. the circle ABT :: the circle
o 14. 5. EFN. K. *o* therefore the circle ABT ⊏ K. *p* and
p 11. 5. EGq. ACq :: the circle EFN. K. which is ſhewn to be repugnant.

Therefore it muſt be concluded, that I is = to the circle EFN. *Which was to be dem.*

Coroll.

Hence it follows, that as a circle is to a circle, ſo is a polygonon deſcribed in one to a like polygonon deſcribed in the other.

PROP. III.



Every Pyramide ABDC having a triangular baſe, may be divided into two Pyramides AEGH, HIKC, equal, and like one to the other, having baſes triangular, and like to the whole ABDC; and into two equal Priſms, BFGEIH, EGDHIK; which two Priſms are greater than the half of the whole Pyramide ABDC.

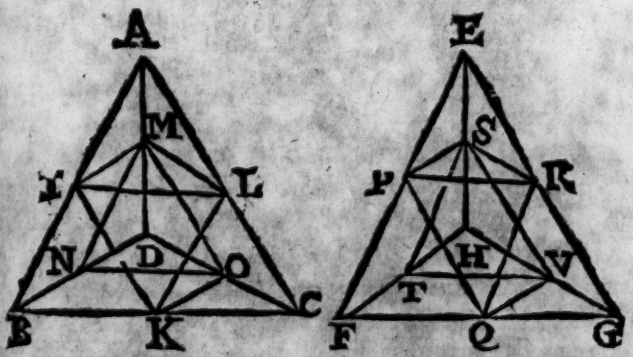
Divide the ſides of the pyramide into two parts at the points E, F, G, H, I, K, and join the right lines EF, FG, GE, EI, IF, FK, KG, GH, HE. Becauſe the ſides of the pyramide are proportionally

cut,

cut, *a* thence HI,AB; and GF, AB; and IF,DC; and HG, DC, *&c.* are parallels, and conſequently HI,FG; and GH,FI are alſo parallels. therefore it is apparent that the triangles ABD,AEG,EBF, FDG, HIK, *b* are equiangular, and that the four laſt are *e* equal: in like manner the triangles ACB,AHE,EIB,HIC,FGK are equiangular; and the four laſt are equal one to the other. Alſo the triangles BFI, FDK, IKC, EGH; and laſtly, the triangles AHG, GDK, HKC, EFI are like and equal. Moreover the triangles HIK to ADB, and EGH to BDC, and EFI to ADC, and FGK to ABC, *d* are parallel. From whence it evidently follows, firſt that the pyramides AEGH, HIKC are equal, and *e* like to the whole ABDC, and to one another. Next, that the ſolids BFGEIH, FGDIHK, are priſms, and that of equal heighth, as being placed between the parallel planes ABD, HIK, but the baſe BFGE is *f* double of the baſe FDG, wherefore the ſaid priſms are equal; whereof the one BFGEIH is greater than the pyramide BFEI, that is, then AEGH, the whole than its part; and conſequently the two priſms are greater than the two pyramides, and ſo exceed the half of the whole pyramide ABDC. *Which was to be dem.*

a 2. 6.
b 29. 1.
c 26. 1.
d 15. 11.
e 10. *def.* 11.
f 2. *ax.* 1.
g 40. 11.

PROP. IV.



If there be two pyramides ABCD, EFGH, of the ſame altitude, having triangular baſes ABC, EFG;

and

and either of them be divided into two pyramides (AILM, MNOD; and EPRS, STVH) equal one to the other, and like to the whole, and into two equal prisms (IBKLMN, KLCNMO; and PFQRST, QRGTSV;) and if in like manner either of those pyramides made by the former division be divided, and this be done continually; then as the base of one pyramide is to the base of the other pyramide, so are all the prisms which are in one pyramide, to all the prisms which are in the other pyramide, being equal in multitude.

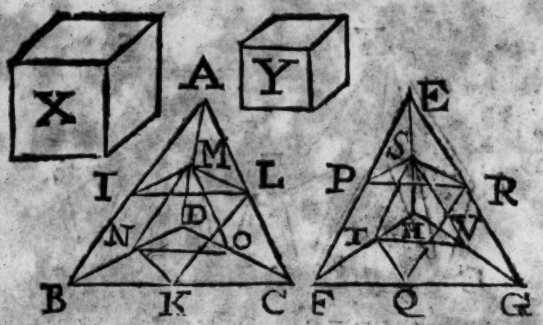
a 15. 5. b 22. 6. c 2. 6. & c. d 16. 5. e *sch.* 34. 11. f 7. 5. g 12. 5.

For (applying the construction of the precedent Prop.) BC. KC *a* :: FG. QG. *b* therefore the triangle ABC is to the like triangle LKC as EFG is to *c* the like RQG. therefore by permutation ABC EFG *d* :: LKC. RQG *e* :: the prism KLCNMO. QRGTSV (for these are of equal altitude) *f* :: IBKLMN. PFQRST. *g* wherefore the triangle ABC. EFG :: the prism KLCMNO + IBKLMN. the prism QRGISV + PFQRST. *Which was to be dem.*

But if the pyramides MNOD, AILM, and EPRS, STVH, be further divided, in like manner the four new prisms made hereby shall be to the four produced before, as the bases MNO and AIL are to the bases STV, and EPR; that is, as LKC to RQG, or as ABC to EFG. *h* wherefore all the prisms of the pyramide ABCD are to all the prisms of the pyramide EFGH as the base ABC is to the base EFG.

PROP.

PROP. V.



Pyramides ABCD, EFGH, being under the ſame altitude, having triangular baſes ABC, EFG, are one to another as their baſes ABC, EFG, are.

Let the triangles ABC. EFG :: ABCD. X. I ſay X is equal to the pyramide EFGH. For if it be poſſible, let X be ⌐ EFGH. and let the exceſs be Y, divide the pyramide EFGH into priſms and pyramides. and the other pyramides in like manner, *a* till the pyramides left EPRS, SVTH, be leſs than the ſolid Y. Therefore ſince the pyramide EFGH = X + Y, it is manifeſt that the remaining priſms PFQRST, QRGTSV are greater than the ſolid X. Conceive the pyramide ABCD divided after the ſame manner; *b* then will be the priſm IBKLMN + KLCNMO. PFQRST + QRGTSV :: ABC. EFG *c* :: the pyr. ABCD. X. *d* therefore X ⊏ the priſm PFQRST + QRGTSV; which is contrary to that which was affirmed before.

a 1. 12.

b 4. 12.

c *hyp.*

d 14. 5.

Again, conceive X ⊏ the pyr. EFGH. and make the pyr. EFGH. Y :: X the pyr. ABCD *e* :: EFG. ABC. Becauſe EFGH *f* ⌐ X, *g* thence Y ⌐ the pyr. ABCD. which is ſhewn before to be impoſſible. Therefore I conclude, that X is equal to the pyr. EFGH. *Which was to be dem.*

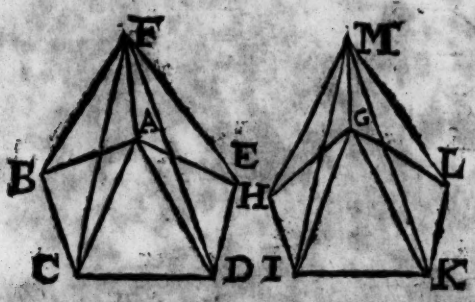
e *hyp. and cor.* 4. 5.

f *ſuppoſ.*

g 14. 5.

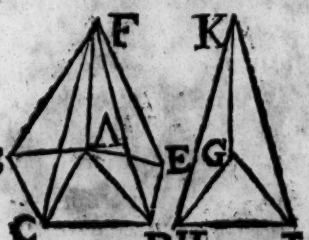
PROP.

PROP. VI.



Pyramides ABCDEF, GHIKLM, consisting under the same altitude, and having polygonous bases ABCDE, GHIKL, are to one another as their bases ABCDE, FGHIKL are.

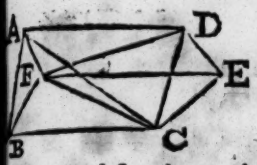
Draw the right lines AC,AD,GI,GK. then is
a 5. 12. the base ABC.ACD *a* :: the pyr. ABCF.ACDF.
b 18. 5. *b* therefore by composition ABCD. ACD :: the pyr. ABCDF. ACDF. *a* but also ACD. ADE ::
c 22. 5. the pyr. ACDF. ADEF. *c* therefore of equality ABCD. ADE :: ABCDF. ADEF. and *b* thence by composition ABCDE. ADE :: the pyr. ABC-
d 5. 12. DEF. ADEF. moreover ADE. GKL *d* :: the pyr. ADEF. GKLM; and as before, and inversely GKL. GHIKL :: the pyr. GKLM. GHIKLM. *c* therefore again of equality ABCDE.GHIKL :: the pyr. ABCDEF. GHIKLM. *Which was to be dem.*



If the bases have not sides of equal multitude, the demonstration will proceed thus.
a 5. 12. The base ABC. GHI *e* :: the pyr. ABCF. GHIK. *e* and ACD. GHI :: the pyr. ACDF. GHIK. *f* therefore the base ABCD. GHI :: the pyr.
f 4. 5. ABCDF. GHIK. *e* Moreover the base ADE. GHI :: the pyr. ADEF. GHIK. *f* therefore the base ABCDE. GHI :: the pyr. ABCDEF. GHIK.

PROP.

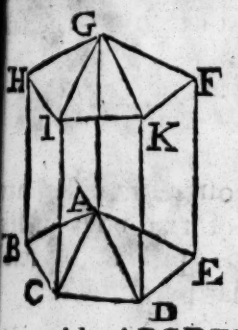
PROP. VII.



Every Priſm, ABCDEF, having a triangular baſe, may be divided into three Pyramides ACBF, ACDF, CDFE, equal one to the other, and having triangular baſes.

Draw the diameters of the parallelograms, AC, CF, FD. Then the triangle ACB is *a* = ACD. *b* therefore the pyramides of equal heighth ACBF. ACDF. are equal. In like manner the pyr. DFAC = the pyr. DFEC. but ACDF and DFAC are one and the ſame pyramide. *c* therefore the three pyramides ACBF, ACDF, DFEC, into which the Priſm is divided, are equal one to the other. *Which was to be dem.*

a 34. 1. b 5. 11. c 1. *ax.* 1.



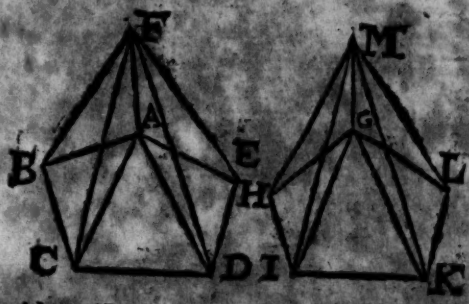
Hence, every pyramide is the third part of the Priſm that has the ſame baſe and heighth with it; or every priſm is treble of the pyramide that has the ſame baſe and heighth with it.

For reſolve the polygonous priſm ABCDEGHIKF into triangular Priſms; and the pyramide ABCDEH into triangular pyramides; *a* then all the parts of the priſm ſhall be treble to all the parts of the pyramide, *b* conſequently the whole priſm ABCDEGHIKF is treble to the whole pyramide ABCDEH. *Which was to be dem.*

a 7. 12. b 1. 5.

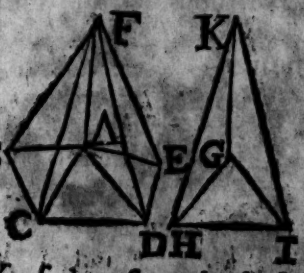
PROP.

PROP. VI.



Pyramides ABCDEF, GHIKLM, consisting under the same altitude, and having polygonous bases ABCDE, GHIKL, are to one another as their bases ABCDE, FGHIKL are.

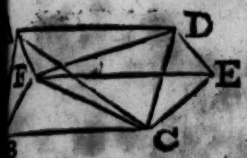
Draw the right lines AC, AD, GI, GK. then is
a 5. 12. the base ABC. ACD *a* :: the pyr. ABCF. ACDF.
b 18. 5. *b* therefore by composition ABCD. ACD :: the pyr. ABCDF. ACDF. *a* but also ACD. ADE ::
c 22. 5. the pyr. ACDF. ADEF. *c* therefore of equality ABCD. ADE :: ABCDF. ADEF. and *b* thence by composition ABCDE. ADE :: the pyr. ABC-
d 5. 12. DEF. ADEF. moreover ADE. GKL *d* :: the pyr. ADEF. GKLM; and as before, and inversely GKL. GHIKL :: the pyr. GKLM. GHIKLM. *c* therefore again of equality ABCDE. GHIKL :: the pyr. ABCDEF. GHIKLM. *Which was to be dem.*



If the bases have not sides of equal multitude, the demonstration will proceed thus.
a 5. 12. The base ABC. GHI *e* :: the pyr. ABCF. GHIK. *e* and ACD. GHI :: the pyr. ACDF. GH-
f 4. 5. IK. *f* therefore the base ABCD. GHI :: the pyr. ABCDF. GHIK. *e* Moreover the base ADE. GHI :: the pyr. ADEF. GHIK. *f* therefore the base ABCDE. GHI :: the pyr. ABCDEF. GHIK.

PROP.

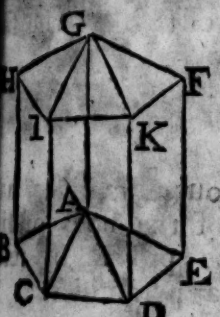
PROP. VII.



Every Prism, ABCDEF, having a triangular base, may be divided into three Pyramides ACBF, ACDF, CDFE, equal one to the other, and having triangular bases.

Draw the diameters of the parallelograms, AC, CF, FD. Then the triangle ACB is *a* = CD. *b* therefore the pyramides of equal heighth CBF. ACDF. are equal. In like manner the pyr. DFAC = the pyr. DFEC. but ACDF and DFAC are one and the same pyramide. *c* therefore the three pyramides ACBF, ACDF, DFEC, into which the Prism is divided, are equal one to the other. *Which was to be dem.*

a 34. 1.
b 5. 11.
c 1. *ax.* 1.



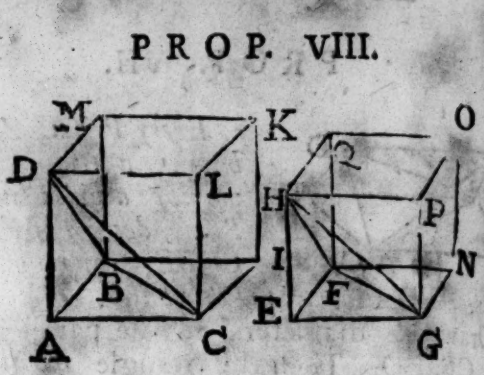
Hence, every pyramide is the third part of the Prism that has the same base and heighth with it; or every prism is treble of the pyramide that has the same base and heighth with it.

For resolve the polygonous prism ABCDEGHIKF into triangular Prisms; and the pyramide ABCDEH into triangular pyramides; then all the parts of the prism shall be treble to all the parts of the pyramide, *b* consequently the whole prism ABCDEGHIKF is treble to the whole pyramide ABCDEH. *Which was to be dem.*

a 7. 12.
b 1. 5.

PROP.

PROP. VIII.



Like Pyramides ABCD, EFGH, which have trian-
gular baſes ABC, EFG are in triple proportion of th
in which their ſides of like proportion AC, EG, are

a 27. 11. *a* Compleat the parallelepipedons ABICDM
b 9. *def.* 11. KL, EFNGHQOP, which *b* are like, and
c 28. 11. ſextuple of the pyramides ABCD, EFGH. *d* an
and 7. 12. therefore in the ſame proportion with them on
d 15. 5. to another, *e* that is, triple of that of the ſide
e 33. 11. of like proportion, *&c.*

Coroll.

Hence, alſo like polygonous pyramides hav
proportion tripled to that of the ſides of lik
proportion; as may eaſily be proved by reſol-
ving the ſame into triangular pyramides.

PROP. IX.

See the preceding Scheme.

In equal pyramides ABCD, EFGH, having trian-
gular baſes ABC, EFG, the baſes and altitudes ar
reciprocal; And pyramides having triangular baſes
whoſe altitudes and baſes are reciprocal, are equal.

1. *Hyp.* The compleated parallelepipedons ABIC-
a 28. 11. DMKL, EFNGHQOP are *a* ſextuple of the equal
and 7. 12. pyramides ABCD, EFGH (either of either) and ſo
equal

equal one to the other. therefore the altitude (H.) the altitude (D) *b* :: ABIC. EFNG *c* :: ABC. EFG. *Which was to be dem.*

b 34. 11.
c 15. 5.

2. *Hyp.* The altitude (H.) the altitude (D) *d* :: ABC. EFG *e* :: ABIC. EFNG. *f* therefore the parallelepipedons ABICDMKL, EFNGHQOP are equal. *g* consequently also the pyramides ABCD, EFGH being subsextuple of the same, are equal. *Which was to be dem.*

d *hyp.*
e 15. 5.
f 34. 11.
g 6. *ax.* 1.

The same is applicable to polygonous pyramides; for they may also in like manner be reduced to triangulars.

Coroll.

Whatsoever is demonstrated of pyramides in Prop. 6. 8, 9. *does likewise agree to any sort of prisms; seeing they are triple of the pyramides that have the same base and altitude with them.* Therefore

1. The proportion of prisms of equal altitude is the same with that of their bases.

2. The proportion of like prisms is triple of that of the sides of like proportion.

3. Equal prisms have their bases and altitudes reciprocal; and prisms which are so reciprocal, are equal.

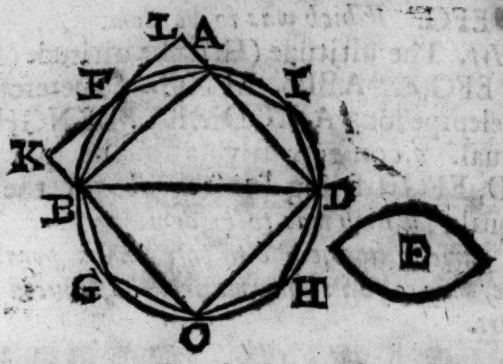
Schol.

From what is hitherto demonstrated the dimension of any prisms and pyramides may be collected.

a The solidity of a prism is produced of the altitude multiplied into the base; *b* and therefore likewise that of a pyramide, of the third part of the altitude multiplied into the base.

a 1. *cor.* 12. & *sch.* 40. 11.
b 7. 12.

PROP. X.



Every Cone is the third part of a cylinder having the ſame baſe with it ABCD, and the altitude equal.

If you deny it, then firſt let ſuch cylinder be more than triple to the cone, and let the exceſs be E. A priſm deſcribed on a ſquare in the circle ABCD *a* is ſubduple of a priſm deſcribed upon a ſquare about the circle, being equal to it and the cylinder in the heighth. Therefore a priſm upon the ſquare ABCD exceeds the half of the cyl. and likewiſe a priſm upon the baſe AFB, of equal heighth to the cylinder, *b* is greater than the half of the ſegment of the cylinder AFB, continue an equal biſection of the arches, and ſubſtract the priſms till the remaining ſegments of the cylinder, namely at AF,FB,*&c.* become leſs than the ſolid E. Therefore the cyl. — ſegm. AF,FB,*&c.* (the priſm on the baſe AFBGCHDI) *c* is greater than the cylinder — E (*d* the triple of the cone.) therefore the pyramide, *e* a third part of the ſaid priſm (being placed on the ſame baſe, and of the ſame heighth) is greater than the cone of equal heighth on the baſe ABCD a circle *i. e.* the part greater than the whole. *Which is abſurd.*

See the ſecond figure of this Book.

a *ſch.* 7. 4. *and cor.* 9. 12.

b *ſch.* 27. 3. *and cor.* 9. 12.

c 5. *ax.* 1.

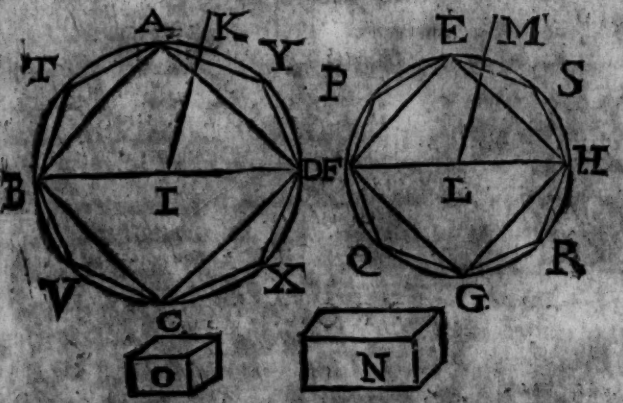
d *hyp.*

e *cor.* 7. 12.

But if the cone be affirmed to be greater than the third part of the cylinder, then let the exceſs be E. Detract the pyramides from the cone, as you did in the firſt part the priſms from the cylinder,

der, till ſome ſegments of the cone remain, conceive at AF, FB, EG, *&c.* leſs than the ſolid E. therefore the cone – E (*f* ⅓ of the cylinder) ⊐ f *hyp.* the pyr. AFBGCHDI (the cone – ſegm. AF, FB, *&c.*) therefore the priſm triple to the pyramide (*viz.* of equal heighth, and on the ſame baſe) is greater than the cylinder, on the baſe ABCD, the part than the whole, *Which is abſurd.* Wherefore it muſt be granted, that the cylinder is equal to triple of the cone.

PROP. XL.



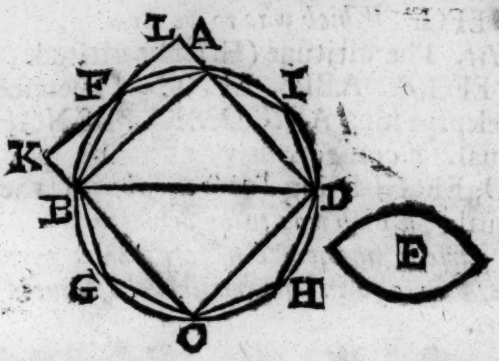
Cylinders and Cones ABCDK, EFGHM, being under the ſame altitude, are to one another as their baſes ABCD, EFGH.

Let the circle ABCD. the cir. EFGH :: the cone ABCDK.N. I ſay N is equal to the cone EFGHM.

For if it be poſſible, let N be ⊐ the cone EFGHM, and let the exceſs be O. The preparation and argumentation of the prec. Prop. being ſuppoſed; then ſhall O be greater than the ſegments of the cone EP, PF, FQ, *&c.* and ſo the ſolid N ⊐ the pyr. EPFQGRHSM. in the circle ABCD, *a* make a like polygonous figure ATBVCXDY. Becauſe the pyr. ABVYK. the pyr. EFQSM *b* :: the pyr. ATBVY. the polyg. EPFQS *c* :: the cir. ABCD. the cir. EFGH *d* :: the cone ABCDK.N. *e* thence the pyr. EPFQGRHSM ſhall be ⊐ N. contrary to what was affirmed before. Again conceive N

a 2 3 0.3. *and* 1. *poſt.*
b 6. 12.
c *cor.* 2. 22.
d *hyp.*
e 14. 5.

PROP. X.



Every Cone is the third part of a cylinder having the ſame baſe with it ABCD, and the altitude equal.

If you deny it, then firſt let ſuch cylinder be more than triple to the cone, and let the exceſs be E. A priſm deſcribed on a ſquare in the circle ABCD *a* is ſubduple of a priſm deſcribed upon a ſquare about the circle, being equal to it and the cylinder in the heighth. Therefore a priſm upon the ſquare ABCD exceeds the half of the cyl. and likewiſe a priſm upon the baſe AFB, of equal heighth to the cylinder, *b* is greater than the half of the ſegment of the cylinder AFB, continue an equal biſection of the arches, and ſubſtract the priſms till the remaining ſegments of the cylinder, namely at AF,FB,&c. become leſs than the ſolid E. Therefore the cyl. — ſegm. AF,FB,&c. (the priſm on the baſe AFBGCHDI) *c* is greater than the cylinder — E (*d* the triple of the cone.) therefore the pyramide, *e* a third part of the ſaid priſm (being placed on the ſame baſe, and of the ſame heighth) is greater than the cone of equal heighth on the baſe ABCD a circle *i. e.* the part greater than the whole. *Which is abſurd.*

See the ſecond figure of this Book.

a *ſch.* 7. 4. *and cor.* 9. 12.

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c 5. *ax.* 1.

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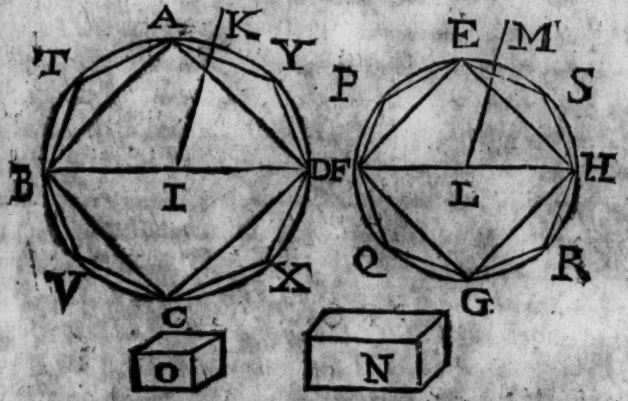
e *cor.* 7. 12.

But if the cone be affirmed to be greater than the third part of the cylinder, then let the exceſs be E. Detract the pyramides from the cone, as you did in the firſt part the priſms from the cylinder,

der, till ſome ſegments of the cone remain, conceive at AF, FB, EG, *&c.* leſs than the ſolid E. therefore the cone – E (*f* ! of the cylinder) ⊐ the pyr. AFBGCHDI (the cone – ſegm. AF, FB, *&c.*) therefore the priſm triple to the pyramide (*viz.* of equal heighth, and on the ſame baſe) is greater than the cylinder, on the baſe ABCD, the part than the whole, *Which is abſurd.* Wherefore it muſt be granted, that the cylinder is equal to triple of the cone. f *hyp.*

PROP. XL.



Cylinders and Cones ABCDK, EFGHM, being under the ſame altitude, are to one another as their baſes ABCD, EFGH.

Let the circle ABCD. the cir. EFGH :: the cone ABCDK.N. I ſay N is equal to the cone EFGHM.

For if it be poſſible, let N be ⊐ the cone EFGHM, and let the exceſs be O. The preparation and argumentation of the prec. Prop. being ſuppoſed; then ſhall O be greater than the ſegments of the cone EP, PF, FQ, *&c.* and ſo the ſolid N ⊐ the pyr. EPFQGRHSM. in the circle ABCD, *a* a3 0.3. *and* make a like polygonous figure ATBVCXDY. Be 1. *poſt.* cauſe the pyr. ABVYK. the pyr. EFQSM *b* :: b 6. 12. ATBVY. the polyg. EPFQS *c* :: the cir. ABCD. c *cor.* 2. 22. the cir. EFGH *d* :: the cone ABCDK.N. *e* thence d *hyp.* the pyr. EPFQGRHSM ſhall be ⊐ N. contrary e 14. 5. to what was affirmed before. Again conceive N

the cone EFGHM. and make the cone EFGH. f *hyp. and by inversion.* MO :: N. the cone ABCDK *f* :: the circle EF. GH. ABCD. *g* therefore O ⊐ the cone ABCDK; *which is absurd*, as appears by what is shewn in g 14. 5. the first part.

Therefore rather admit ABCD. EFGH :: the cone ABCDK. EFGHM. *Which was to be dem.*

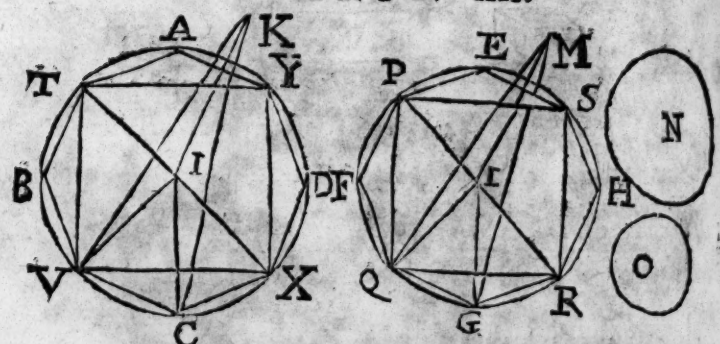
The same may be demonstrated of cylinders, if cylinders and prisms be conceived in the place of cones and pyramides. therefore, *&c.*

Schol.

Hence is gathered the dimension of all sorts of cylinders and cones. The solidity of a right cylin- a 1. *Prop. de dimens. circ.* der is produced of the circular base (*a* the dimension whereof is to be learnt out of *Archimedes*) multiplied into the heighth ; *b* whence b 11. 12. in like manner that of every cylinder.

Therefore the solidity of a cone is produced of the third part of the altitude multiplied into the base.

PROP. XII.



Like cones and cylinders ABCDK, EFGHM, are in triplicate proportion of that of the diameters TX, PR, of their bases ABCD, EFGH.

Let the cone A have to N triplicate proportion of TX to PR. I say N is = the cone EFGHM. For if it be possible, let N be ⊐ EFGHM. and let the excess be O. therefore, N ⊐ the pyr. EPFQGRHSM. Let the axes of the cones be IK, LM, and join the right lines VK, CK, VI, CI, and QM, GM,

GM, QL, GL. Becauſe the cones are like, *a* thence VI. IK :: QL. LM. but the angles VIK, QLM *b* are right angles. *c* therefore the triangles VIK, QLM are equiangular. *d* whence VC, VI :: QG. QL. alſo VI. VK :: QL. QM. therefore of equality VC. VK :: QG. QM. *e* moreover VK. GK :: QM. MG. therefore again of equality VC. CK :: QG. GM. *f* therefore the triangle VKC, QMG are like: and by the ſame reaſon the other triangles of this pyramide are like to the other of that; *g* wherefore the pyramides themſelves are like. *h* But they are in triplicate proportion of that of VC to QG. *k* that is, of VI to RL, *l* or TX to PR. *m* therefore the pyr. AIBVCXDYK. the pyr. EPFQGRHSM :: the cone of ABCDK. N. *n* whence the pyr. EPFQ-GRHSM ⊐ N. which is repugnant to what was affirmed before.

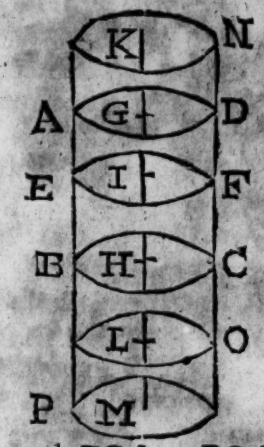
a 24. *def.* 11.
b 18. *def.* 11.
c 6. 6.
d 4. 6.
e 7. 5.
f 5. 6.
g 9. *def.* 11
h *cor.* 8. 12.
k 4. 6.
l 15. 5.
m *hyp. and* 11. 5.
n 14. 5.

Again, take N ⊏ the cone EFGHM. make the cone EFGHM. O :: N. the cone ABCDK *o* :: the pyr. EPRM. ATCK *p* :: GQ. VC thrice :: *q* PR. TX thrice. but O *r* is ⊐ ABCDK. Wherefore N = the cone EFGHM. *Which was to be dem.*

o *before, and inverſely.*
p *cor.* 8. 12.
q 4. 6.
r 14. 5.

But foraſmuch as what proportion ſoever cones have, alſo cylinders being triple of them, have the ſame; therefore cylinder to cylinder ſhall have proportion triplicate of the diameters of the baſes.

PROP. XIII.

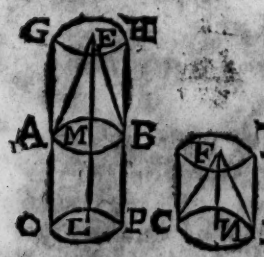


If a cylinder ABCD, be divided by a plane EF parallel to the oppoſite planes BC, AD, then as one cylinder AFED is to the other cylinder EBCF, ſo is the axis GI to the axis HI.

The axis being produced, a take GK = GI, and HL = IH = LM, and conceive planes drawn at the points K, L, M, parallel to the circles AD, BC, b therefore the cylinder FD = the cyl. AN. and the cyl. EC b = BO b = OP. therefore the cylinder EN is as multiple of the cylinder ED as the axis IK is of the axis IG. and in like manner the cylinder FP is as multiple of the cylinder BF, as the axis IM is of the axis IH. but as IK is =, ⊏, ⊐ IM, c ſo is the cylinder EN =, ⊏, ⊐ EP. d therefore the cylinder AEFD. the cyl. EBCF :: GI. IH. *Which was to be dem.*

a 3. 1.
b 11. 12.
c 11. 12.
d 6. *def.* 5.

PROP. XIV.



Cones AEB, CFD, and cylinders AH, CK conſiſting upon equal baſes AB, CD, are to one another as their altitudes ME, NF.

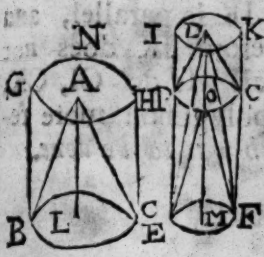
The cylinder AH, and the axis EM being produced, take ML = FN ; and at the point L draw a plane parallel to the baſe AB, a then ſhall the cyl. AP be = CK. b but the cyl. AH. AP(CK) :: ME. ML (NP.) *Which was to be dem.*

* *apply* 9. *and* 7. 12. The ſame may be affirmed of cones ſubtriple of cylinders ; * as alſo of priſms and pyramides.

a 11. 12.
b 13. 12.

PROP.

PROP. XV.



In equal cones BAC, EDF, and cylinders BH, EK, the baſes and altitudes are reciprocal BC. EF :: MD. LA.) And cones and cylinders, whoſe baſes and altitudes are reciprocal, are equal one to another.

If the altitudes be equal then the baſes are equal too, and the thing is evident. If unequal, then take away MO = LA.

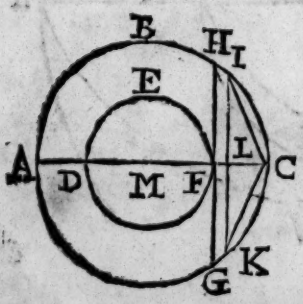
1. *Hyp.* Then is MD. MO (*a* LA) *b* :: the cyl. EK. (*c* BH) EQ *d* :: the cir. BC. EF. *Which was to be dem.*

2. *Hyp.* BC. EF *e* :: DM. OM (LA) *f* :: the cyl. EK. FQ *g* :: BC. EF *h* :: BH. EQ. Therefore the cylinder EK = BH. *Which was to be dem.*

The ſame argument may be uſed for cones.

a 14. 12.
b *conſtr.*
c *hyp.*
d 11. 12.
e *hyp.*
f 11. 12.
g 11. 5.
h 11. 12.
k 9. 5.

PROP. XVI.



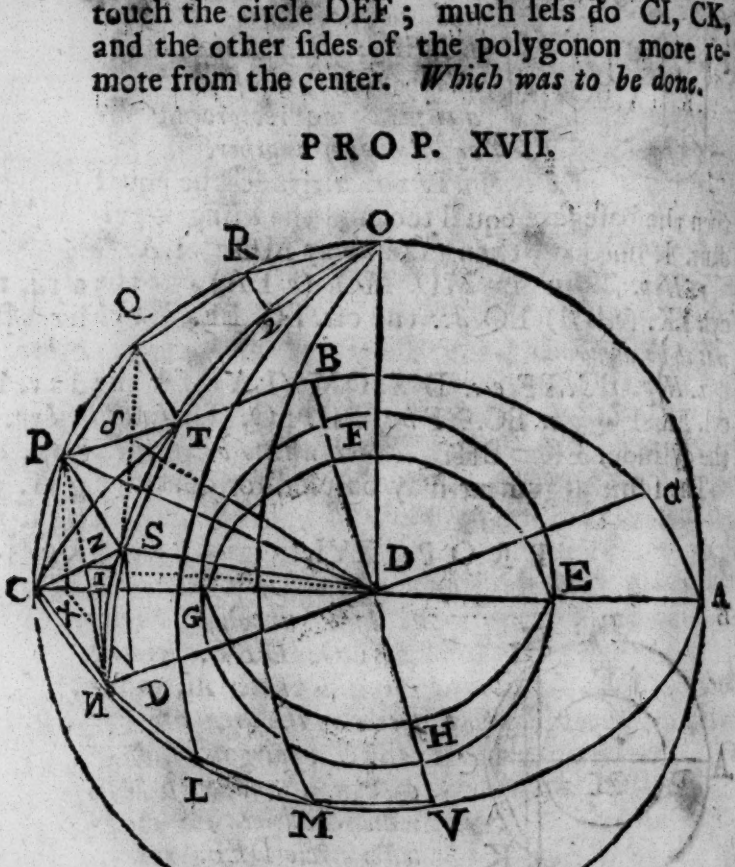
Two unequal circles ABCG, DEF, having the ſame center M, to inſcribe in the greater circle ABCG a polygonous figure of equal, and even ſides, which ſhall not touch the leſſer circle DEF.

Through the center M draw the line AC cutting the circle DEF in F, from whence raiſe a perpendicular FH. *a* divide the ſemicircle ABC into two equal parts; and the half thereof BC alſo; and ſo do continually. *b* till the arch IC become leſs than the arch HC, from I let fall the perpendicular IL. It is manifeſt that the arch IC meaſures the whole circle, and that the number of arches is even, and

a 30. 3.
b 1. 10.

c *sch.*16.4. so that the subtended line IC is the side of the polygon that may be inscribed without touch-
d *cor.*16.3. ing the lesser circle DEF. For HG *d* touches
e 28. 1. the circle DEF. *e* to which IK is parallel, and
f 34.*def.*1. placed outwardly; *f* wherefore IK does not touch the circle DEF; much less do CI, CK, and the other sides of the polygonon more remote from the center. *Which was to be done.*

PROP. XVII.



Two spheres ABCV, EFGH, consisting about the same center D, being given, to inscribe a solid of many sides (or Polyedron) in the greater sphere ABCV, which shall not touch the superficies of the lesser sphere EFGH.

Let

Let both the ſpheres be cut by a plane paſſing by the center making the circles EFGH, ABCV; and the diameters AC, BV drawn, cutting perpendicularly. In the circle ABCV, *b* inſcribe the equilateral polygone VMLNC, *&c.* not touching the circle EFGH: then draw the diameter N*a*, and erect DO perpendicular to the plane ABC. by DO, and by the diameters AC, N*a*, conceive planes DOC, DON erected, which ſhall be *b* perpendicular to the circle ABCV, and ſo in the ſuperficies of the ſphere make *c* the quadrants DOC, DON. In which let the right lines CP, PQ, QR, RO, NS, ST. Tγ, γO *d* be fitted, equal, and of equal multitude with CN, NL, *&c.* make the ſame conſtruction in the other quadrants OL, OM, *&c.* and in the whole ſphere. Then I ſay the thing required is done.

From the points P, S, to the Plane ABCV draw the perpendiculars PX, SY, *e* which ſhall fall on the ſections ACN*a*. Therefore becauſe both *f* the right angles PXC, SYN, *g* and PCX, SNY inſiſting on *h* equal circumferences, *f* are equal, the triangles alſo PCX, SNY *h* are equiangular. Wherefore being PC *k* = SN, *l* alſo is PX = SY, *l* and XC = YN; *m* whence DX = DY. *n* and therefore DX. XC :: DY. YN. *o* therefore YX, NC are parallels. but becauſe PX, SY are equal, and ſince being perpendicular to the ſame plane ABCV, they are alſo *p* parallels, *q* therefore YX, SP ſhall be equal and parallel. *r* whence SP, NC, are parallel one to the other; and ſo the *ſ* quadrilateral NCPS. and by the ſame reaſon SPQT, TQRG, as alſo the *t* triangle γ RO are ſo many planes. In like manner the whole ſphere may be ſhewn full of ſuch quadrilaterals and triangles. wherefore the figure inſcribed is a polyedron.

From the center D*u* draw DZ perpendicular to the plane NCPS; and join ZN, ZC, ZS, ZP. Becauſe DN. NC *x* :: DY. YX. thence NC is *y* = YX

(SP)

a 16. 12.
b 18. 11.
c *cor.* 33. 6.
d 4. 1.
e 38. 11.
f 12. *ax.*
g 27. 3.
h 32. 1.
k *conſtr.*
l 26. 1.
m 3. *ax.* 1.
n 7. 5.
o 2. 6.
p 6. 11.
q 33. 1.
r 9. 12.
ſ 7. 11.
t 2. 11.
u 11. 11.
x 4. 6.
y 14. 5.

(SP.) and likewiſe SP ⊏ TQ, and TQ ⊏ γ R. And becauſe the angles DZC,DZN,DZS,DZP
z 3.*def.*11. *z* are right, and the ſides DC,DN,DS,DP, *a* equal,
a 15. *def.* 11. and DZ common, *b* thence ZC,ZN,ZS,ZP are equal one to the other; and conſequently about
b 47. 1. the quadrilateral NCPS *c* a circle may be deſcri-
c 15.*def.*1. bed, in which (becauſe NS,NC,CP, are *d* equal,
d *conſtr.* and NC ⊏ SP) NC *e* ſubtends more than the quadrant, *f* therefore the ang. NZC at the center is
e 28. 3. obtuſe, *g* therefore NCq ⊏ 2 ZCq (ZCq+ZNq.)
f 33. 9. Let NI be drawn perpendicular to AC. there-
g 12. 2. fore ſince the angle ADN (*h* DNC+DCN) *k* is
h 32. 1. obtuſe, the half of it DCN ſhall be greater than
k 9. *ax.* 1. the half of a right angle; and ſo that which remains of the right ang. CNI ſhall be leſs than it
l 5. 1. *n* whence IN ⊏ IC. therefore NCq (NIq+ICq)
n 19. 1. *o* ⊐ INq. therefore IN ⊏ ZC. and conſequently
o 47. 1. DZ *p* ⊏ DI. but the point I is *q* without the ſphere
p 47. 1. EFGH. and ſo much more the point Z. wherefore
q *cor.* 19. the plane NCPS, (whoſe *r* next point to the cen-
12. ter is Z) does not touch the ſphere EFGH. And
r 47. 1. if a perpendicular D ♌ be drawn to the plane SPQT, the point ♌, and ſo alſo the plane SPQT is yet further removed from the center, which is alſo true of the other planes of the polyedron. Therefore the polyedron ORPQCN, *&c.* inſcribed in the greater ſphere, does not touch the leſſer. *Which was to be done.*

Coroll.

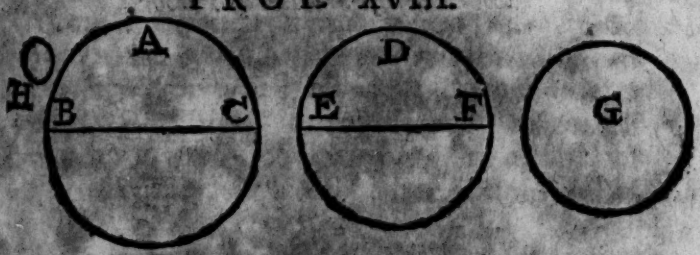
Hence it follows, *That if in any other ſphere a ſolid polyedron, like to the aboveſaid ſolid polyedron, be inſcribed, the proportion of the polyedron in one ſphere to the polyedron in the other is triplicate of that of the diameters of the ſpheres.*

For if right lines be drawn from the centers of the ſpheres to all the angles of the baſes of the ſaid polyedrons, then the polyedrons will be divided into pyramides equal in number and like; whoſe

whose homologous sides are semidiameters of the spheres; as appears, if the lesser of these spheres be conceived described within the greater about the same center. For the right lines drawn from the center of the sphere to the angles of the bases will agree one to the other by reason of the likeness of the bases; and so will like pyramids be made. Wherefore since every pyramid in one sphere to every pyramid like it in the other sphere, *a* has proportion triple to that of the homologous sides, that is, of the semidiameters of the spheres; and *b* as one pyramide is to one pyramide, so all the pyramides, that is, the solid polyedron composed of these, are to all the pyramides, that is, the solid polyedron composed of the others; therefore the polyedron of one space shall have to the polyedron of the other sphere, proportion triple of that of the semidiameters, *c* and so of the diameters of the spheres.

a *cor.* 8. 12.
b 12. 5.
c 15. 5.

PROP. XVIII.



Spheres BAC, EDF, are in triplicate proportion one to the other of that in which their diameters BC, EF.

Let the sphere BAC be to the sphere G in triple proportion of that of the diameter BC to the diameter EF. I say G = DEF. For if it be possible, let G be ⊐ EDF. and conceive the sphere G concentrical with EDF. In the sphere of EDF *a* inscribe a polyedron not touching the sphere G, and a like polyedron in the sphere BAC.

a 17. 12.

(SP.) and likewiſe SP ⊏ TQ, and TQ ⊏ γ R. And becauſe the angles DZC,DZN,DZS,DZP
z 3.*def.*11. z are right, and the ſides DC,DN,DS,DP, *a* equal,
a 15. *def.* and DZ common, *b* thence ZC,ZN,ZS,ZP are e-
11. qual one to the other; and conſequently about
b 47. 1. the quadrilateral NCPS *c* a circle may be deſcri-
c 15.*def.*1. bed, in which (becauſe NS,NC,CP, are *d* equal,
d *conſtr.* and NC ⊏ SP) NC *e* ſubtends more than the qua-
e 28. 3. drant, *f* therefore the ang. NZC at the center is
f 33. 9. obtuſe, *g* therefore NCq ⊏ 2 ZCq (ZCq+ZNq.)
g 12. 2. Let NI be drawn perpendicular to AC. there-
h 32. 1. fore ſince the angle ADN (*b* DNC+DCN) *k* is
k 9. *ax.* 1. obtuſe, the half of it DCN ſhall be greater than
l 5. 1. the half of a right angle; and ſo that which re-
mains of the right ang. CNI ſhall be leſs than it
n 19. 1. *n* whence IN ⊏ IC. therefore NCq (NIq + ICq)
o 47. 1. *o* ⊐ INq. therefore IN ⊏ ZC. and conſequently
p 47. 1. DZ *p* ⊏ DI. but the point I is *q* without the ſphere
q *cor.* 19. EFGH. and ſo much more the point Z. wherefore
12. the plane NCPS, (whoſe *r* next point to the cen-
r 47. 1. ter is Z) does not touch the ſphere EFGH. And if a perpendicular D ♌ be drawn to the plane SPQT, the point ♌, and ſo alſo the plane SPQT is yet further removed from the center, which is alſo true of the other planes of the polyedron. Therefore the polyedron ORPQCN, *&c.* inſcribed in the greater ſphere, does not touch the leſſer. *Which was to be done.*

Coroll.

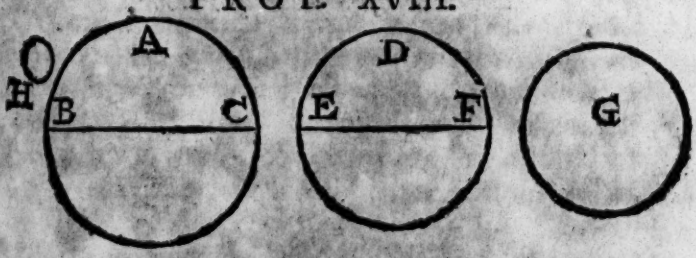
Hence it follows, *That if in any other ſphere a ſolid polyedron, like to the aboveſaid ſolid polyedron, be inſcribed, the proportion of the polyedron in one ſphere to the polyedron in the other is triplicate of that of the diameters of the ſpheres.*

For if right lines be drawn from the centers of the ſpheres to all the angles of the baſes of the ſaid polyedrons, then the polyedrons will be divided into pyramides equal in number and like; whoſe

whoſe homologous ſides are ſemidiameters of the ſpheres; as appears, if the leſſer of theſe ſpheres be conceived deſcribed within the greater about the ſame center. For the right lines drawn from the center of the ſphere to the angles of the baſes will agree one to the other by reaſon of the likeneſs of the baſes; and ſo will like pyramides be made. Wherefore ſince every pyramide in one ſphere to every pyramide like it in the other ſphere, *a* has proportion triple to that of the homologous ſides, that is, of the ſemidiameters of the ſpheres; and *b* as one pyramide is to one pyramide, ſo all the pyramides, that is, the ſolid polyedron compoſed of theſe, are to all the pyramides, that is, the ſolid polyedron compoſed of the others; therefore the polyedron of one ſpace ſhall have to the polyedron of the other ſphere, proportion triple of that of the ſemidiameters, *c* and ſo of the diameters of the ſpheres.

a *cor.* 8. 12.
b 12. 5.
c 15. 5.

PROP. XVIII.



Spheres BAC, EDF, are in triplicate proportion one to the other of that in which their diameters BC, EF.

Let the ſphere BAC be to the ſphere G in triple proportion of that of the diameter BC to the diameter EF. I ſay G = DEF. For if it be poſſible, let G be ⊐ EDF. and conceive the ſphere G concentrical with EDF. In the ſphere of EDF *a* inſcribe a polyedron not touching the ſphere G, and a like polyedron in the ſphere BAC.

a 17. 12.

b *cor.* 17. 12. BAC. These polyedrons *b* are in triplicate proportion of the diameters BC, EF. *c* that is, of the
c *hyp.* sphere BAC to G. *d* Consequently the sphere
d 14. 5. G is greater than the polyedron inscribed in the sphere EDF, the part than the whole.

Again, if it be possible, let the sphere G be ⊏ EDF. and as the sphere EDF is to another
e *hyp. inverſ.* sphere H, so let G be to BAC, *e* that is, in proportion of the diameter EF to BC. there-
f 14. 5. fore since BAC *f* ⊏ H, we shall incur the absurdity of the first part. wherefore rather the sphere G = EDF. *Which was to be dem.*

Coroll.

Hence, As one sphere is to another sphere, so is a Polyedron described in that, to a like polyedron described in this.

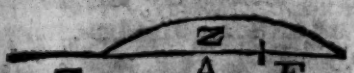
The End of the twelfth Book.

THE

THE THIRTEENTH BOOK OF EUCLIDE's ELEMENTS.

PROP. I.

IF a right line z *be divided according to extreme and mean proportion* (z. a :: a. e.) *the ſquare of the half of the whole line* z, *and of the greater ſegment* a, *as one line is quintuple to that which is deſcribed of half of that whole line* z.



I ſay Q. a + ½ z = 5 Q: ½ z. *a* that is, aa + ¼ zz + za = zz + ¼ zz. *b* or aa + za = zz. For ze + za *c* = zz. and ze *d* = aa. *e* therefore aa + za = zz. *Which was to be dem.*

a 4. 2.
b 3. *ax.* 1.
c 2. 2.
d *hyp. and* 16. 6.
e z. *ax. and* 1. *ax.*

PROP. II.

See the 1. *Scheme.*

If a right line ½ z + a *be in power quintuple to a ſegment of it ſelf* ½ z. *the line double of the ſaid ſegment* (z) *being divided according to extreme and mean proportion, the greater ſegment is* (a) *the other part of the right line at firſt given* ½ z + a.

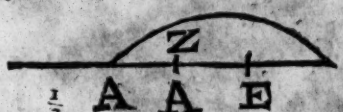
I ſay z. a :: a. e. For becauſe by the hyp. * aa + ¼ zz + za = zz + ¼ zz; or aa + za = zz *a* = ze + za, *b* thence ſhall aa be = ze. *c* wherefore z. a :: a. e. *Which was to be dem.*

* 4. 2.
a 2. 2.
b 3. *ax.* 1.
c 17. 6.

PROP.

PROP. III.

If a right line z *be divided according to extreme and mean proportion* (z. a :: a. e.) *the line made of the leſs ſegment* e *and half of the greater ſegment* a, *is in power quintuple to the ſquare, which is deſcribed of the half line of the greater ſegment* a.

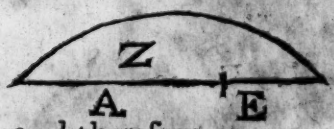


I ſay Q: e + ½ a = 5 Q: ½ a. *a* that is ee + ½ aa + ea = aa + ¼ aa. *b* or ee + ea = aa. For ee + ea *c* = ze *d* = aa. *Which was to be dem.*

a 4. 2.
b 3. *ax.*
c 3. 2.
d *hyp. and* 17. 6.

PROP. IV.

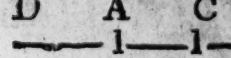
If a right line z *be cut according to extreme and mean proportion* (z. a :: a. e.) *the ſquare made of the whole line* z, *and that made of the leſſer ſegment* e, *both together, are triple of the ſquare made of the greater ſegment* a.



I ſay zz + ee = 3 aa. *a* or aa + ee + 2 ae + ee = 3 aa. For ae + ee *b* = ze *c* = aa. *d* therefore aa + 2 ae + 2 ee = 3 aa. *Which was to be dem.*

a 4. 1.
b 3. 2.
c 17. 6.
d 2. *ax.*

PROP. V.



If a right line AB be cut according to extreme and mean proportion in C, *and a line AD, equal to the greater ſegment BC, added to it, the whole right line DB is divided according to extreme and mean proportion; and the greater ſegment the right line AB given at the beginning.*

For becauſe AB. AD *a* :: AC. CB. and by inverſion AD. AB :: CB. AC. therefore by compoſition DB. AB :: AB, AC (AD.) *Which was to be dem.*

a *hyp.*

Schol.

Schol.

But if BD. BA :: BA. AD. then ſhall be BA. AD :: AD. BA — AD. For by diviſion is BD — BA (AD.) BA :: BA — AD. AD. therefore inverſely BA. AD :: AD. BA — AD.

PROP. VI.

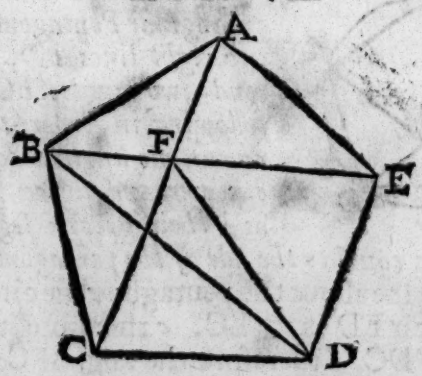
D A C B

If a rational right line AB be cut according to extreme and mean proportion in C, either of the ſegments (AC, CB) is an irrational line of that kind which is called apotome or reſidual.

To the greater ſegment AC *a* add AD = ½ AB. *b* therefore DCq = 5 DAq. *c* therefore DCq □ DAq. conſequently *d* ſince AB, *e* and ſo the half thereof DA are ῥ, likewiſe DC is ῥ. But becauſe 5. 1 :: not Q. Q. *f* thence is DC □ DA. *g* therefore DC — AD, that is, AC, is a reſidual line. Further, becauſe ACq *h* = AB × BC, and AB is ῥ. *k* likewiſe EC is a reſidual line. *Which was to be dem.*

a 3. 1.
b 1. 13.
c 6. 10.
d *hyp.*
e *ſch.* 12. 10.
f 9. 10.
g 74. 10.
h 17. 6.
k 98. 10.

PROP. VII.



If three angles of an equilateral Pentagone ABCDE whether they follow in order, (EAB, ABC, BCD,) or not, (EAB, BCD, CDE) be equal, the Pentagone ABCDE ſhall be equiangular.

Let

Let the right lines BE, AC, BD, be ſubtended to the equal angles in order.

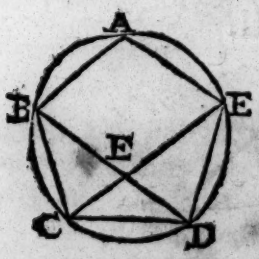
Being the ſides EA, AB, BC, CD, and the included angles *a* are equal, therefore ſhall the baſes BE, AC, BD, *c* and the angles AEB, ABE, BAC, BCA, be equal. *d* wherefore BF = FA. *e* and conſequently FC = FE; therefore the triangles FCD, FED, are equilateral one to the other: *f* whence the angle FCD = FED. *g* conſequently the angle AED = BCD. In like manner the ang. CDE is equal to the reſt; wherefore the pentagone is equiangular. *Which was to be dem.*

a *hyp.* b 4. 1. c 4. *and* 5. 1. d 6. 1. e 3. *ax.* 1. f 8. 1. g 2. *ax.* 1.

But if the angles EAB, BCD, CDE, which are not in order, be ſuppoſed equal. *b* then ſhall the angle AEB be = BDC, and BE = BD. *k* and thence the angle BED = BDE. *l* conſequently the whole angle AED = CDE. therefore becauſe the angles A, E, D, in order, are equal, as before, the pentagone ſhall be equiangular. *Which was to be dem.*

h 4. 1. k 5. 1. l 2. *ax.*

PROP. VIII.



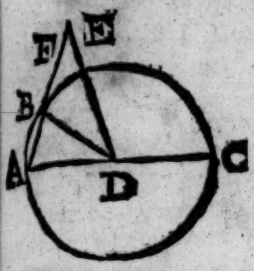
If in an equilateral and equiangular Pentagone ABCDE, two right lines BD, CE, ſubtend two angles BCD, CDE, following in order, thoſe lines do cut one another according to extreme and mean proportion, and their greater ſegments BF or EF are equal to the ſide of the pentagone BC.

a Deſcribe about the pentagone the circle ABD. *b* The arch ED is = BC, *c* therefore the angle FCD = FDC, *d* therefore the ang. BFC = 2 FCD (FCD + FDC. But the arch BAE *b* is = 2 ED, and conſequently the angle BCF *e* = 2 FCD = BFC. *f* wherefore BF = BC. *Which was to be dem.* Moreover becauſe the triangles BCD, FCD, are *g* equiangular, *h* therefore BD. DC (BF.) :: CD. (BF)

a 14. 4. b 28. 3. c 27. 3. d 32. 1. e 33. 6. f 6. 1. g 27. 3. h 4. 6.

(BF.) FD. and likewiſe EC.EF :: EF. FC. *Which was to be dem.*

PROP. IX.



If the ſide of an Hexagone BE, and the ſide of a Decagone AB, both deſcribed in the ſame circle ABC, be added together, the whole right line AE is cut according to extreme and mean proportion (AE. BE :: BE. AB.) and the greater ſegment thereof is the ſide of the Hexagone BE.

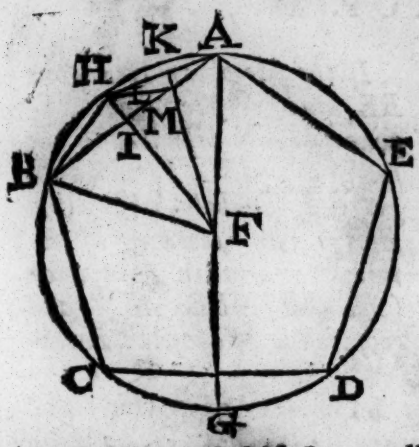
Draw the diameter ADC, and join the right lines DB, DE. Becauſe the angle BDC *a* = 2 BDA. and the angle BDC *b* = 2 DBA (DAC + DBA) thence ſhall DBA (*b* DBE + BED) *c* be = 2 BDA *d* = 2 BDE. whence the angle DBA or DAB *e* = ADE. Therefore the triangles ADE, ADB, are equiangular: *f* wherefore AE. AD (*g* BE) :: AD. (BE.) AB. *Which was to be dem.*

a *hyp.* and 27. 3.
b 32. 1.
c 7. *ax.* 1.
d 5. 1.
e 1. *ax.* 1.
f 4. 6.
g *cor.* 15. 4.

Coroll.

Hence, If the ſide of an Hexagone in a circle be cut according to extreme and mean proportion; the greater ſegment thereof ſhall be the ſide of the Decagone in the ſame circle.

PROP. X.



If an equilateral Pentagone ABCDE be described in a circle ABCE, the side of the Pentagone AB contains in power both the side of a hexagone FB, and the side of a decagone AH described in the same circle.

Draw the diameter AG. and biſect equally the arch AH in K. and draw FK, FH, FB, BH, HM.

a 28. 3. *and* 3. *ax.* The ſemicircle AG — the arch AC *a* = AG — AD. that is, the arch CG = GD *b* = AH = HB. therefore the arch BCG = 2 BHK; *c* and ſo the angle BFG = 2 BFK. *d* but the angle BFG = 2 BAG. *e* therefore the angle BFK = BAG. Wherefore the triangles BFM, FAB, *f* are equiangular. *g* whence AB. BF :: BF. KM. *h* therefore AB × BM = BFq. Moreover, the angle AFK *k* = HFK, and FA = FH. *m* wherefore AL = LH. *m* and the angles FLA, FLH are equal, and ſo right angles. therefore the angle LHM *m* = LAM *n* = HBA. therefore the triangles AHB, AMH, *o* are equiangular. wherefore AB. AH :: AH. AM. *q* therefore AB × AM = AHq. So that ſeeing ABq *r* = AB × BM + AB × AM, *ſ* thence ABq = BFq + AHq. *Which was to be dem.*

b *hyp. and* 7. *ax.*
c 33 6.
d 20. 3.
e 1. *ax.* 1.
f 31. 1.
g 4. 6.
k 27. 3.
m 4. 1.
n 27. 3.
o 32. 1.
p 4. 6.
q 17. 6.
r 2. 2.
ſ 2. *ax.*

Coroll.

1. Hence, a right line (FK) which being drawn from the center (F) divides an arch (HA) into two equal ſegments, does alſo divide the right line

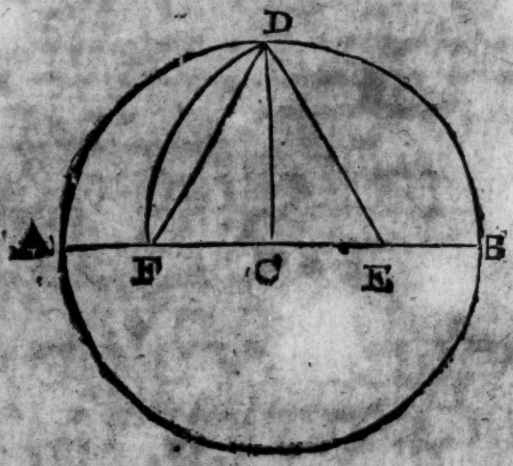
ſine (HA) ſubtending that arch, perpendicularly into two equal ſegments.

2. The diameter of a circle (AG) drawn from any angle (A) of a pentagone, does divide equally in two both the arch (CD) which the ſide of the pentagone oppoſite to that angle ſubtends, and alſo the oppoſite ſide it ſelf (CD) and that perpendicularly.

Schol.

Here, according to our Promiſe, we ſhall lay down a ready praxis of the 11. *Prop. of the* 4. *Book.*

Problem.



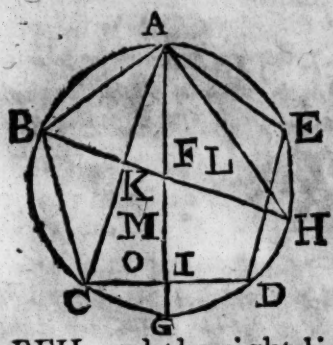
To find out the ſide of a pentagone to be inſcribed in a circle ADB.

Draw the diameter AB, to which erect a perpendicular CD at the center C divide CB equally in E. and make EF = ED. then DF ſhall be the ſide of the pentagone. a 6. 2.

For BF × FC + ECq *a* = EFq *b* = EDq *c* = DCq + ECq. *d* therefore BF × FC + DCq or BCq. *e* wherefore BF. BC :: BC. FC. therefore ſince BC is the ſide of an hexagone, *f* FC ſhall be the ſide of a decagone. Conſequently DF *h* = √ DCq + FCq *g* is the ſide of a pentagone. *Which was to be done.*

b *conſtr.*
c 47. 1.
d 3. *ax.*
e 17. 6.
f 9. 13.
g 10. 13.
h 47. 1.

PROP. XI.



If in a circle ABCD, whose diameter is rational AG, an equilateral pentagone be inscribed ABCDE; the side of the pentagone AB is an irrational line of that kind which is called a minor line.

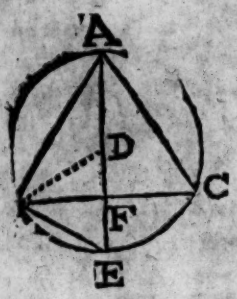
Draw the diameter BFH, and the right lines AC, AH; and * make FL = $\frac{1}{4}$ of the radius FH; and CM = $\frac{1}{4}$ CA.

Because the angles AKF, AIC, are *a* right angles, and CAI common, the triangles AKF, AIC, are *b* equiangular: *c* therefore CI. FK *c* :: CA. FA (FB) *d* :: CM. FL. therefore by permutation FK. FL :: CI. CM *d* :: CD. CK (2 CM) and so by *e* composition CD + CK. CK :: KL. FL. *f* consequently ◻ CD + CK (*g* 5 CKq.) CKq :: KLq. FLq. therefore KLq = 5 FLq. wherefore if BH (ρ) be taken 8, FH shall be 4, FL 1, and FLq 1, BL 5, and BLq 25, KLq 5, by which it appears that BL and KL are ρ *h* ⫟, *k* and so BK is a residual, and KL its congruent or adjoining line, but being BLq − KLq = 20, *l* thence BL ⫍ √ BLq − KLq. *m* whence BK shall be a fourth residual line. Therefore because ABq *m* is = HB × BK, *n* shall AB be a minor line. *Which was to be dem.*

* 10. 6.
a *cor.* 10. 13.
b 32. 1.
c 4. 6.
d 15. 5.
e 18. 5.
f 22. 6.
g 1. 13.
h 9. 10.
k 74. 10.
l 9. 10.
m *cor.* 8. 6. *and* 17. 6.
n 95. 10.

PROP.

PROP. XII.



If in a circle ABEC an equilateral triangle ABC be inſcribed, the ſide of that triangle AB is in power triple to the line AD drawn from D the center of the circle to the circumference.

The diameter being extended to E, draw BE. Becauſe the arch BE *a* = EC, the arch BE is the ſixth part of the circumference, *b* therefore BE = DE. hence AEq *c* = 4 DEq (4 BEq) *d* = ABq + BEq (+ ADq.) *e* conſequently ABq = 3 ADq. *Which was to be dem.*

a *cor.* 10. 13.
b *cor.* 15. 4.
c 4. 2.
d 47. 1,
e 3. *ax.* 1.

Coroll.

1. AEq. ABq :: 4. 3.

2. ABq. AFq :: 4. 3. *f* For ABq. AFq :: AEq. ABq.

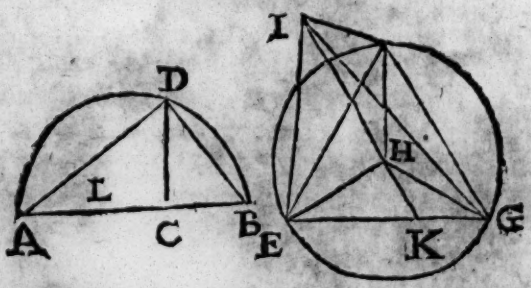
f *cor.* 8. 6. *and* 22. 6.

3. DF = FE. For the triangle EBD *g* is equilateral, *h* and BF perpendicular to ED. *h* therefore EF = FD.

g *cor.* 13. 4.
h *cor.* 3. 3.

4. Hence, AF = DE + DF = 3 DF.

PROP. XIII.



To deſcribe a pyramide EGFI, and comprehend it in a ſphere given: and to demonſtrate that the diameter of the ſphere AB is in power ſeſquialter of the ſide EF of the pyramide EGFI.

a 10. 6. About AB deſcribe the ſemicircle ADB; *a* and let AC be = 2 CB. from the point C erect the perpendicular line CD; and join AD, DB, then at the interval of the radius HE = CD deſcribe
b *cor.* 15. 4. the circle HEFG, wherein inſcribe the equilate-
c 12. 11. ral triangle EFG from H *c* erect IH = CA perpen-
d 3. 1. dicular to the plane EFG. produce IH to K, *d* ſo that IK = AB; and join the right lines IE, IF, IG. Then EFGI ſhall be the pyramide required.

For becauſe the angles ACD, IHE, IHF, IHG,
e *conſtr.* *e* are right angles; and CD, HE, HF, HG *e* equal.
f 41. 1. *e* and IH = AC; *f* therefore AD, IE, IF, IG
g 20. 6. ſhall be equal among themſelves. But being AC
h 2. *ax.* (2 CB) CB *g* :: ACq. CDq. then ſhall ACq be =
k 12. 13. 2 CDq. therefore ADq *f* = ACq + CDq *h* =
l 1. *ax.* 1 3 CDq = 3 HEq *k* = EFq. *l* therefore AD, EF, IE, IF, IG are equal, and ſo the pyramide EF. GI is equilateral. But if the point C be placed upon H, and AC upon HI, the right lines AB,
m 8. *ax.* IH, *m* ſhall agree, as being equal. Wherefore the ſemicircle ADB being drawn about the axis
n 15. *def.* AB or IK *n* ſhall paſs by the point E, F, G,
* 31. *def.* and ſo the pyramide EFGI ſhall be inſcribed in
11. a ſphere. *Which was to be done.*
o *cor.* 8. 6. Alſo it is manifeſt that BAq. ADq *o* :: BA.
p *conſtr.* AC *p* :: 3. 2. *Which was to be dem.*

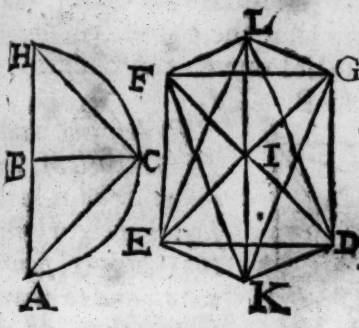
Coroll.

1. ABq. HEq :: 9. 2. For if ABq be put 9, then ACq (EFq) ſhall be 9, conſequently HEq
q 12. 13. ſhall be 2.

2. If L be the center, then ſhall AB. LC :: 6. 1. For if AB be put 6, then AL ſhall be 3. *r* and thence AC 4. wherefore LC ſhall be 1. Hence
r *conſtr.* 3. AB. HI :: 6. 4 :: 3. 2. whence
4. ABq. HIq :: 9. 4.

PROP.

PROP. XIV.



To deſcribe an Octaedron KEFGDL, and comprehend it in the given ſphere, wherein a Pyramide is: and to demonſtrate that AH the diameter of the ſphere is in power double of AC the ſide of that Octaedron.

About H deſcribe the ſemicircle ACH. and from the center B erect the perpendicular BC. draw AC, HC, then upon ED=AC. *a* make the ſquare EFGD, whoſe diameters DF, EG, cut in the center I. from I draw IL=AB *b* perpendicular to the plane EFGD. produce IL, *c* till IK=IL. and join KE, KF, KG, KD, LE, LF, LG, LD; then ſhall KEFGDL be the Octaedron required.

a 46. 1.

b 12. 11.

c 3. 1.

For AB, BH, FI, IE, *&c.* being ſemidiameters of equal ſquares are equal one to the other. *d* whence the baſes LF, LE, FE, *&c.* of the right-angled triangles LIE, LIF, FIE, *&c.* are equal, and conſequently the eight triangles LFE, LEG, LGD, LDE, KEF, KFG, KGD, KDE, are equilateral, *e* and make an Octaedron, which may be inſcribed in a ſphere, whoſe center is I, and IL or AB the radius. (becauſe AB, IL, IF, IK, *&c.* *f* are equal.) *Which was to be done.* Moreover, it is evident that AHq (LKq) *g* = 2 ACq (2 LDq.) *Which was to be dem.*

d 4. 1.

e 27. *def.* 11.

f *conſtr.*

g 47. 1.

Coroll.

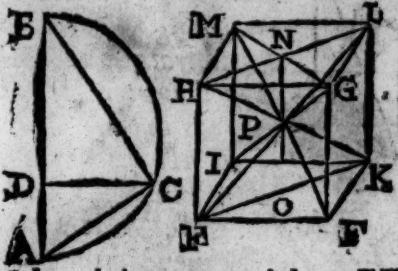
1. Hence it is manifeſt, that in the octaedron the three diameters, EG, FD, LK do cut one the other perpendicularly in the center of the ſphere.

2. Alſo that the three planes EFGD, LEKG, LFKD are ſquares, cutting one another perpendicularly.

3. The Octaedron is divided into two like and equal pyramides EFGDL, and EFGDK, whose common base is the square EFGD.

15. 11. 4. Lastly, It follows that the opposite bases of the Octaedron are parallel one to the other.

PROP. XV.



To describe a cube EFGHIKLM, and comprehend it in the same sphere wherein the former figures were; and to demonstrate that AB the diameter of the sphere is in power triple to EF the side of that cube.

a 10. 6. Upon AB describe a semicircle ACB; *a* and make AB=3 DA, from D raise the perpendicular DC, and join BC and AC. Then upon EF=AC b 46. 11. *b* make the square EFGH, upon whose plane let the right lines EI, FK, HM, GL, stand perpendicular, being equal to EF, and connect them with the right lines IK, KL, LM, IM. The solid EFGHIKLM is a cube, as is sufficiently apparent from the construction.

In the opposite squares EFKI, HGLM, draw the diameters EK, FI, HL, MG, by which let the planes EKLH, FI, MG be drawn, cutting one another in the line NO. which *c* shall divide equally in two parts the diameters of the cube EL, FM, GI, HK, in P the center of the cube, *d* therefore P shall be the center of a sphere passing by the angular points of the cube. Moreover, ELq *e*=EKq +KLq, *e* = 3 KLq, *f* or 3 ACq, but ABq. ACq *g*:: BA. DA *f*:: 3. 1. *h* therefore AB = EL. wherefore we have made a cube, *&c. Which was to be done.*

c *cor.* 39. 11.
d 15. *def.* 11. *and* 14. *def.* 11.
e 47. 1.
f *constr.*
g *cor.* 8. 6.
h 14. 5.

Coroll.

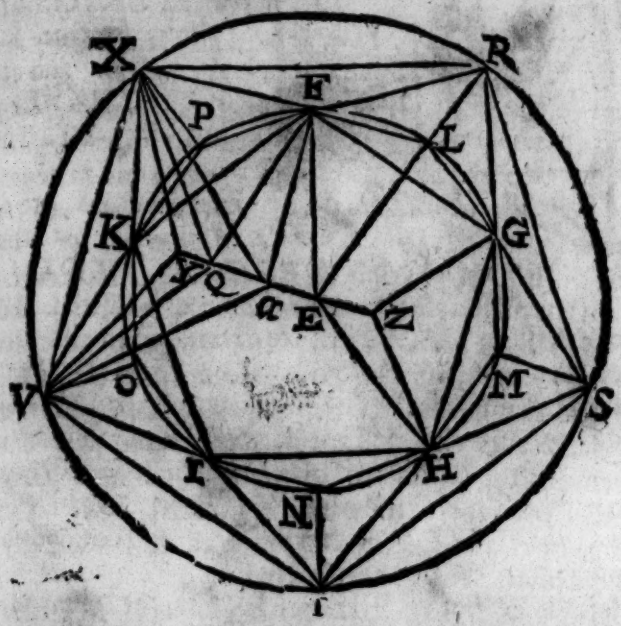
1. Hence it is manifest, that all the diameters of the cube are equal one to another, and do equally bisect one another in the center of the sphere. And by the same means the right lines which

which conjoin the centers of the oppoſite ſquares are equally biſected in the ſame center.

2. The diameter of a ſphere containsin power the ſide of a tetraedron and of a cube, *viz.* ABq $k = l$ BCq $+ m$ ACq.

k 47. 1.
l 13. 15.
m 15. 13.

PROP. XVI.



To deſcribe an Icoſaedron ZGHIKFYVXRST, *and encompaſs it in the ſphere, wherein were contained the foreſaid ſolids; and to demonſtrate that* FG *the ſide of the Icoſaedron is that irrational line which is called a minor line.*

Upon AB the diameter of a ſphere deſcribe the ſemicircle ADB; and *a* make AB $=$ 5 BC. then from C erect CD perpendicular, and draw AD and BD. At the diſtance EF $=$ BD deſcribe the circle EFKNG;

a 10. 6.

b 11. 4. NG; *b* wherein inſcribe the equilateral pentagone FKIHG. Divide equally in two parts the arches FG,GH,*&c.* and join the right lines FL,LG,*&c.*
c 12. 11. being the ſides of a decagone. Then *c* erect EQ, LR,MS,NT,OV,PX equal to EF, and perpendicular to the plane FKNG; and connect RS,ST, TV, VX, XR; as alſo FX, FR, GR, GS, HS, ST, HT, IT, IV, KV, KX. Laſtly, produce FQ, and take QY = FL, and EZ = FL. and conceive the right lines ZG, ZH, ZI, ZK, ZF to be drawn; as alſo YV, YX, YR, YS, YT. Then I ſay the Icoſaedron required is made.

d *conſtr.* For becauſe EQ, LR, MS, NT, OV, PX, are *d*-
e 6. 11. equal and *e* parallel, alſo thoſe lines that join them
f 33. 1. EL, QR, EM, QS, EN, QT, EO, QV, EP, QX, *f* are equal and parallel. And thence likewiſe LM (or FG) RS, MN, ST, *&c.* are equal one to the other.
g 15. 11. *g* therefore the plane drawn by EL, EM, *&c.* is
h 1. *def.* 3. equidiſtant from the plane paſſing by QR, QS, *&c.* *h* and the circle QXRSTV drawn from the center Q is equal to the circle EPLMNO; and, RSTVX is an equilateral pentagone. But EF, EG, EH, *&c.* and QX, QR, QS, *&c.* being conceived to
k 47. 1. be drawn; then becauſe FRq *k* = FEq + LRq, *l* or
l *conſtr.* EFq *m* = FGq, *n* therefore FR, FG, and ſo all
m 10. 13. RS, FG, FR, RG, GS, GH, *&c.* ſhall be equal one
n *ſch.* 48. 1. to the other, and conſequently the ten triangles
and 1. *ax.* RFX, RFG, RGS, *&c.* are equilateral and equal.
o *cor* 14. 11. Moreover, becauſe XQY is a *o* right angle, therefore XYq *p* = QXq + QYq *q* = VXq or FGq.
p 47. 1. wherefore XY, VX. and likewiſe YV, YT, YS, YR,
q 10. 13. ZG, ZH, *&c.* are equal. Therefore other ten triangles are made equilateral and equal both one to another, and to the ten former; and ſo an Icoſaedron is made.

Moreover, divide equally EQ in α, draw the
r 15. *def.* 1. right lines αF, αX, αV; and becauſe QX *r* = QV, and αQ the common ſide, and FQX, EQV are
s 4. 1. right angles, *f* therefore ſhall αX be = αV; and by the ſame reaſon all the lines αX, αR, αS, αT, αV,

αV, αF, αG, αH, αI, αK are equal. But becauſe t ZQ.QE t :: QE.ZE. therefore $Z\alpha q$ $u = 5$ $E\alpha qu$ $x = EQq$ (EFq) + $E\alpha q$ $y = \alpha Fq$. therefore $Z\alpha = \alpha F$. z in like manner $\alpha F = Y\alpha$. therefore the ſphere, whoſe center is α, and αF the ray, ſhall paſs by the 12 angular points of the Icoſaedron.

t 9. 13. u 3. 13. x 4. 2. y 47. 1. z 15. 5.

Laſtly, Becauſe $Z\alpha$. αE :: ZY. QE; a and ſo $Z\alpha q$. αEq :: ZYq. QEq, b therefore ZYq $= 5$ QEq, or 5 BDq $=$: but ABq. BDq c :: AB.BC :: 5. 1. d therefore ZY $=$ AB. *Which was to be done.*

a 22. 6. b 14. 5. c *cor.* 8. 6. d 1. *ax.* 1.

Therefore if AB be put ρ, e then EF $= \sqrt{}$ ABq ſhall be alſo ρ. and conſequently FG the ſide of the pentagone, and likewiſe of the Icoſaedron, ff is a minor line. *Which was to be dem.*

e *ſch.* 12. f 11. 13.

Coroll.

1. From hence is inferred, that the diameter of the ſphere is in power quintuple of the ſemidiameter of the circle encompaſſing the five ſides of the Icoſaedron.

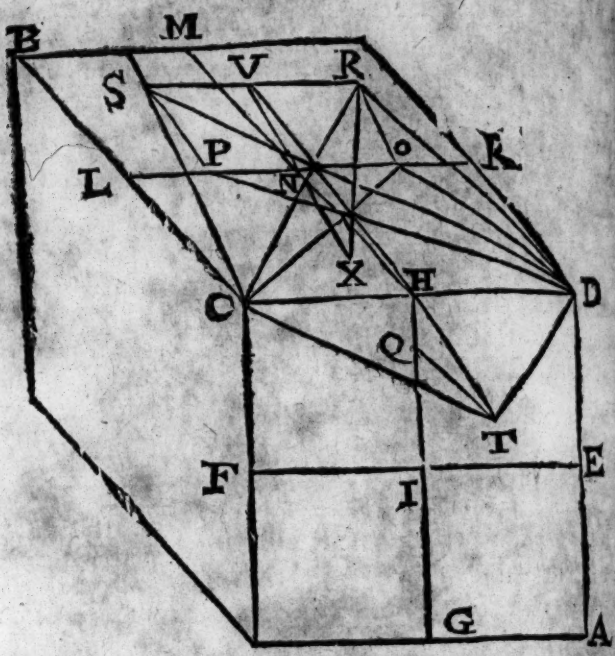
2. Alſo it is manifeſt that the diameter of the ſphere is compoſed of the ſide of a hexagone, that is, of the ſemidiameter, and two ſides of the decagone of a circle encompaſſing the five ſides of the Icoſaedron.

3. It appears likewiſe that the oppoſite ſides of an Icoſaedron, ſuch as RX, HI, are parallels. For RX a is parall. to LP. b parall. to HI.

a 33. 1. b *ſch.* 26. 3.

PROP.

PROP. XVII.



To describe a Dodecaedron, and comprehend it in the sphere wherein the former figures were comprehended: and to demonstrate that the side RS of the Dodecaedron is an irrational line of that sort which is called an Apotome or residual line.

Let AB be a cube inscribed in the given sphere, and let all the sides thereof be divided equally in the points E,H,F,G,K,L,*&c.* and join the right
* 30. 6. lines KL,MH,HG,EF. *a* make HI.IQ::IQ.QH; and take NO, NP,=IQ. then erect OR, PS, perpendicular to the plane DB, and QT to the plane AC, and let OR, PS, QT, be equal to IQ, NO, NP, whence DR,RS,SC,CT,DT, being connected, DRSCT shall be a pentagone of the dodecaedron required. For draw NV parallel to OR, and having drawn NV out as far as the center of the cube X, join the right lines DS, DO, DP, CR, CP,

CP,HV,HT,RX. Becauſe DOq *a* = DKq (*b* KNq) a 47. 1.
+KOq *c* = 3 ONq (3 ORq) *d* thence DRq = b 7. *ax.* 1.
3 ORq *e* = OPq, or RSq. therefore DR = RS. c 4. 13.
By the ſame reaſon DR,RS,SC,CT, TP are equal. d 47. 1.
But becauſe OR *f* is = and *g* parallel to PS, e 4. 2.
therefore RS,OP, and conſequently RS, DC ſhall l *conſtr.* 96.
be alſo parallels. *h* therefore theſe with them 11.
that conjoin them DK,CS,VH, are in one and the g 33. 1.
ſame plane. Moreover, becauſe HI. IQ *h* :: IQ. h 9. 1.
(TQ.) QH *k* :: HN. NV. and both TQ,HN, and k 7. 11.
QH,NV *k* are perpendicular to the ſame plane, *l* l 6. 11.
and ſo likewiſe parallels, *m* THV ſhall be a right m 32. 6.
line. *n* therefore the Trapezium DRSC, and the n 1. *and* 2.
triangle DTS are in one plane extended by the 11.
right lines DC,TV. *o* therefore DCTSR is a pen- o 5. 13.
tagone, and that alſo equilateral, by what is
ſhewn already. Furthermore, becauſe PK. KN ::
KN.NP; and DSq *p* = DPq + PSq (PNq) = *p* p 47. 1.
DKq+PKq+NPq, *q* thence DSq = DKq + 3 KNq q 1. *ax.* 1.
= 4 DKq (4 DHq) *r* = DCq. therefore DS = DC. *and* 4. 13.
whence the triangles DRS, DCT, are equilateral r 4. 2.
one to another, *ſ* therefore the angle DRS = DTC, ſ 8. 1.
therefore the pentagone DTCSR is alſo equian-
gular. Moreover, becauſe AX, DX, CX, *&c.* are
ſemidiameters of the cube. *t* thence is XN = IH t 15. 13.
or KN, *u* and ſo XV = KP; wherefore becauſe u 1. *ax.* 1.
RVX, is a *x* right angle, *z* thence RXq = XVq + x 29. 1.
RVq (NPq) = KPq + NPq *a* = 3 KNq *b* = AXq z 47. 1.
or DXq, *&c.* therefore RX, AX, DX, and by the a 4. 13.
ſame reaſon XS,XT,AX, are equal one to another. b 15. 13.
And if by the ſame method, whereby the penta-
gone DTCSR was made, twelve like penta-
gones, touching the twelve ſides of the cube, be
made, they ſhall compoſe a Dodecaedron; and a
ſphere paſſing by their angular points, whoſe
radius is AX or RX, ſhall comprehend that Do-
decaedron. *Which was to be done.*

Laſtly, becauſe KN.NO *c* :: NO. OK, *d* thence c *conſtr.*
KL.OP :: OP.OK + PL. Therefore if AB the dia- d 15. 5.
meter of the ſphere be ſuppoſed *p*, then ſhall

KL

e 15. 13. f *ſch.* 12. 10. g 6. 13. KL $e = \sqrt{\frac{AB}{3}}$ *f* be alſo ῥ. *g* whence OP or RS the ſide of the dodecaedron ſhall be a reſidual line. *Which was to be dem.*

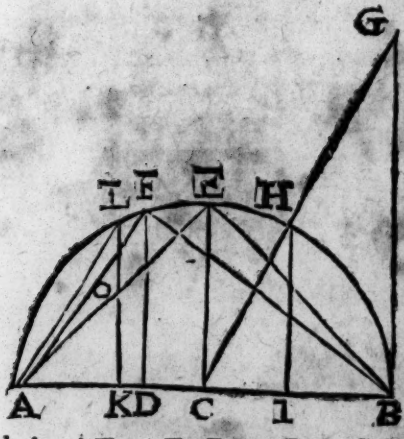
Coroll.

From this demonſtration it follows, 1. that if the ſide of a cube be cut in extreme and mean proportion, the greater ſegment ſhall be the ſide of the dodecaedron inſcribed in the ſame ſphere.

2. If the leſſer ſegment of a right line, cut in extreme and mean proportion, be the ſide of the dodecaedron, the greater ſegment ſhall be the ſide of the cube inſcribed in the ſame ſphere.

3. It is manifeſt alſo, that the ſide of the cube is equal to the right line which ſubtends the angle of a pentagone of the dodecaedron, inſcribed in the ſame ſphere.

PROP. XVIII.



To find out the ſides of the precedent five figures, and compare them together.

Let AB be the diameter of the ſphere given, and AEB the ſemicircle, and let AC *a* = ⅓ AB, and AD *b* = ⅓ AB then erect the perpendiculars CE, DF, and BG = AB. join AF, AE, BE, BF, CG; and let fall the perpendicular HI from H; and CK being taken equal to CI, from K erect the perpendicular KL, and join AL. Laſtly, *c* make AF. AO :: AO. OF.

a 10. 1. b 10. 6. c 30. 6.

There-

Therefore 3. 2 *d* :: AB. BD *e* :: ABq. BFq the side of a Tetraedron and 2. 1 :: *a* AB. AC :: ABq. BEq. *f* the side of an Octaedron.

Also 3. 1 *d* :: AB. AD *e* :: ABq. AFq. *g* the side of an Hexaedron.

Moreover, because AF. AO *h* :: AO. OF. *k* thence shall AO be the side of a Dodecaedron.

Lastly, BG, (2 BC.) BC *l* :: HI. IC. *m* therefore HI = 2 CI *n* = KI. therefore HIq *o* = 4 CIq. *q* consequently CHq *p* = 5 CIq. *q* therefore ABq = 5 KIq. *r* therefore KI or HI is a radius of a circle enclosing the pentagone of an Icosaedron; and AK or IB *r* is the side of a decagone inscribed in the same circle, *s* whence AL shall be the side of a pentagone, *t* and also the side of an Icosaedron. Whereby it appears that BF, BE, AE are $\dot{\rho}$ ⏉. and AL, AO $\dot{\rho}$ ⏉, and BF ⊏ BE, and BE, AF, and AF ⊏ AO. And because 3 AFq = ABq *u* = 5 KLq, and AF × AO ⊏ AF × OF. *x* and so AF × AO + AF × OF, *y* that is, AFq ⊏ *z* 2 AOq. thence shall 3 AFq (5 KLq) be ⊏ 6 AOq, consequently KL ⊏ AO, and much rather AL ⊏ AO.

That we may express the sides in numbers; If AB be supposed $\surd$ 60, then, reducing what is already shewn to supputation, BF = $\surd$ 40, and BE = $\surd$ 30, and AF $\surd$ 20. Also AL = $\surd$: 30 − $\surd$ 180 (for AK = $\surd$ 15 − $\surd$ 3. and KL (HI) = $\surd$ 12:) Lastly AO = $\surd$: 30 − $\surd$ 500 ($\surd$ 25 − $\surd$ 5.)

Schol.

Schol.

It is very apparent that beſides the five aforeſaid figures, there cannot be deſcribed any other regular ſolid figure (viz. *ſuch as may be contained under ordinate and equal plane figures.*)

For three plane angles at leaſt are required to the conſtituting of a ſolid angle; *a* all which muſt be leſs than four right angles, *b* but 6 angles of an equilateral triangle, 4 of a ſquare, and 6 of a hexagon, do ſeverally equal 4 right angles; and 4 of a pentagon, 3 of a heptagon, 3 of an octagon, *&c.* do exceed 4 right angles: Therefore only of 3, 4, or 5 equilateral triangles, of 3 ſquares or 3 pentagones, it is poſſible to make a ſolid angle. Wherefore beſides the five above-mentioned, there cannot be any other regular bodies.

a 21. 11.
b *See ſchol.* 32. 1.

Out of P. Herigon.

The Proportions of the ſphere and the five regular figures inſcribed in the ſame.

Let the diameter of the ſphere be 2. then ſhall

The Periphery or circumference of the greater circle, be 6, 28318.

The ſuperficies of the greater circle, 3, 14159.

The ſuperficies of the ſphere, 12, 56637.

The ſolidity of the ſphere, 4, 1879.

The ſide of the Tetraedron, 1, 62299.

The

The ſuperficies of the tetraedron, 4, 6188.

The ſolidity of the tetraedron, 0, 15132.

The ſide of the hexaedron, 1, 1547.

The ſuperficies of the Hexaedron, 8.

The ſolidity of the hexaedron, 1, 5396.

The ſide of the Octaedron, 1, 41421.

The ſuperficies of the octaedron, 6, 9282.

The ſolidity of the octaedron, 1, 13333.

The ſide of the Dodecaedron, 0, 71364.

The ſuperficies of the dodecaedron, 10, 51462.

The ſolidity of the dodecaedron, 2, 78516.

The ſide of the Icoſaedron, 1, 05146.

The ſuperficies of the Icoſaedron, 9, 57454.

The ſolidity of the Icoſaedron, 2, 53615.

Schol.

It is very apparent that beſides the five aforeſaid figures, there cannot be deſcribed any other regular ſolid figure (viz. *ſuch as may be contained under ordinate and equal plane figures.*)

For three plane angles at leaſt are required to the conſtituting of a ſolid angle; *a* all which muſt be leſs than four right angles, *b* but 6 angles of an equilateral triangle, 4 of a ſquare, and 6 of a hexagon, do ſeverally equal 4 right angles; and 4 of a pentagon, 3 of a heptagon, 3 of an octagon, *&c.* do exceed 4 right angles: Therefore only of 3, 4, or 5 equilateral triangles, of 3 ſquares or 3 pentagones, it is poſſible to make a ſolid angle. Wherefore beſides the five above-mentioned, there cannot be any other regular bodies.

a 21. 11.
b *See ſchol.* 32. 1.

Out of P. Herigon.

The Proportions of the ſphere and the five regular figures inſcribed in the ſame.

Let the diameter of the ſphere be 2. then ſhall

The Periphery or circumference of the greater circle, be 6, 28318.

The ſuperficies of the greater circle, 3, 14159.

The ſuperficies of the ſphere, 12, 56637.

The ſolidity of the ſphere, 4, 1879.

The ſide of the Tetraedron, 1, 62299.

The

The ſuperficies of the tetraedron, 4, 6188.

The ſolidity of the tetraedron, 0, 15132.

The ſide of the hexaedron, 1, 1547.

The ſuperficies of the Hexaedron, 8.

The ſolidity of the hexaedron, 1, 5396.

The ſide of the Octaedron, 1, 41421.

The ſuperficies of the octaedron, 6, 9282.

The ſolidity of the octaedron, 1, 33333.

The ſide of the Dodecaedron, 0, 71364.

The ſuperficies of the dodecaedron, 10, 51462.

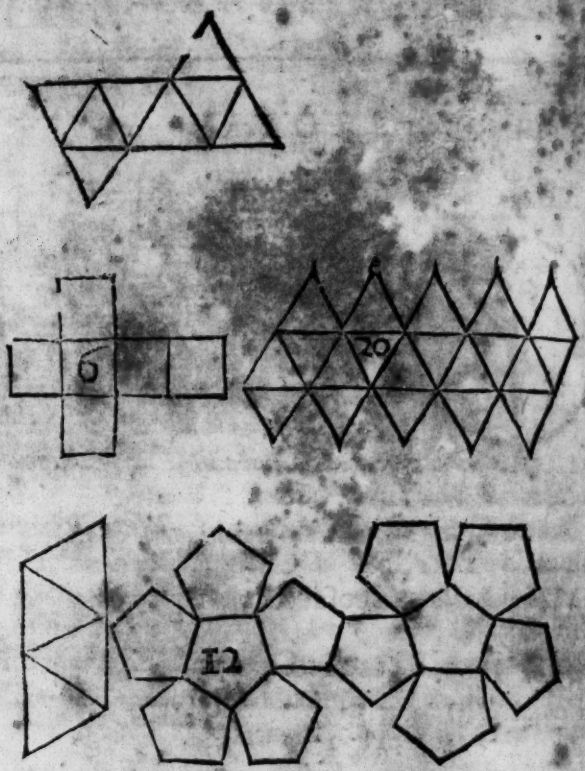
The ſolidity of the dodecaedron, 2, 78516.

The ſide of the Icoſaedron, 1, 05146.

The ſuperficies of the Icoſaedron, 9, 57454.

The ſolidity of the Icoſaedron, 2, 53615.

If five equilateral and equiangular figures, like these in the schemes beneath, be made of Paper, and rightly folded, they will represent the five regular bodies.

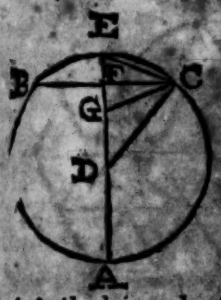


The End of the thirteenth Book.

THE

THE FOURTEENTH BOOK OF EUCLIDE's ELEMENTS.

PROP. I.



A Perpendicular line DF drawn from D the center of a circle ABC to BC the side of a pentagone inscribed in the said circle, is the half of these two lines taken together, viz. of the side of the hexagone DE, and the side of the decagone EC inscribed in the same circle ABC.

Take FG = FE, and draw CG : *a* Then CE is = CG, therefore the angle CGE *b* = CEG *b* = ECD. therefore the angle ECG *c* = EDC *d* = ½ ADC *e* = ½ CED (½ ECD) *f* consequently the angle GCD = ECG = EDC. *g* wherefore DG = GC (CE.) therefore DF = CE (DG) + EF = DE + CE.
2

Which was to be dem.

a 4. 1.
b 5. 1.
c 32. 1.
d *hyp. and* 33. 6.
e 20. 3.
f 7. *ax.*
g 6. 1.

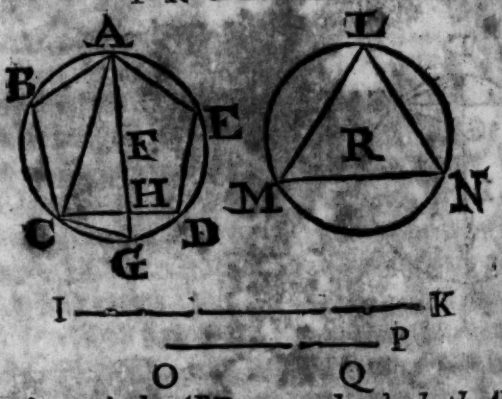
PROP. II.

A G B C
D H E F

If two right lines AB, DE, be cut according to extreme and mean proportion (AB. AG :: AG. GB. and DE. DH :: DH. HE.) they shall be cut after the same manner, viz. into the same proportions (AG. GB :: DH. HE.)

a 17. 6. Take BC = BG; and EF = EH. Then AB × BG
b 8. 2. is *a* = 5 AGq. wherefore ACq *b* = 4 ABG + AGq
c 2. *ax.* 1. *c* = BGq. In like manner ſhall DFq be = 5 DHq.
d 22. 5. & 22. 6. *d* therefore AC. AG :: DF. DH. whence by addition AC + AG. AG :: DF + DH. DH. that
e 22. 5. is, 2 AB. AG :: 2 DE. DH. *e* conſequently AB.
f 17. 5. AG :: DE. DH; *f* whence by diviſion AG. GB :: DH. HE. *Which was to be dem.*

PROP. III.



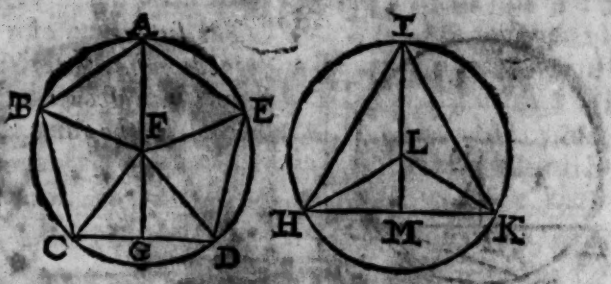
a *ſch.* 47. 1.
b 30. 6.
c 47. 1.
d 4. 2.
e 10. 13.
f 1, *and* 3. *ax.*
g 8. 13.
h 2. 13. & 16. 5.
k 22. 6. & 4. 4.
l 15. 13.
m *conſtr.*
n *cor.* 16. 13.
o 12. 13.
p 10. 13.
q 15. 5.
* *before* 11. *ax.* 1. *and ſch.* 48. 1.
f 1. *def.* 3.

The ſame circle ABD comprehends both ABCDE the pentagone of a Dodecaedron, and LMN the triangle of an Icoſaedron inſcribed in the ſame ſphere.

Draw the diameter AG. and the right lines AC, CG. and let IK be the diameter of the ſphere, *a* and IKq = 5 OPq. *b* and make OP. OQ :: OQ. QP. Becauſe ACq + CGq *c* = AGq *d* = 4 FGq; and ABq *e* = FGq. *f* thence ACq + ABq = 5 FGq. moreover, becauſe CA. AB *g* :: AB. CA — AB; and OP. OQ :: OQ. QP. *h* and ſo CA. OP :: AB. OQ. *k* therefore 3 ACq (*l* IKq) 5 OPq (*m* IKq) :: 3 ABq. 5 OQq. therefore 3 ABq = 5 OQq. But becauſe ML *n* is the ſide of a pentagone inſcribed in a circle, whoſe radius is OP, thence 15 RMq. *o* = 5 MLq *p* = 5 OPq + 5 OQq = * 3 ACq + 3 ABq *q* = 15 FGq. *r* therefore RM = FG. *ſ* and conſequently the circle ABD is = to the circle LMN. *Which was to be dem.*

PROP.

PROP. IV.



If from F the center of a circle encompaſſing the pentagone of a dodecaedron ABCDE, a perpendicular line FG be drawn to one ſide of the pentagone CD; the rectangle contained under the ſaid ſide CD and the perpendicular FG, being thirty times taken, is equal to the ſuperficies of a Dodecaedron. Alſo,

If from the center L of a circle incloſing the triangle of an Icoſaedron HIK, a perpendicular line LM be drawn to one ſide of the triangle HK, the rectangle contained under the ſaid HK, and the perpendicular LM, being thirty times taken, ſhall be equal to the ſuperficies of an Icoſaedron.

Draw FA,FB,FC,FD,FE. *a* then ſhall the triangles CFD,DFE,EFA,AFB,BFC be equal. but CD × FG *b* = 2 triangles CFD. therefore 30 CD × GF *c* = 60 CFD *d* = 12 pentagones ABCDE *e* = to the ſuperficies of a dodecaedron. *Which was to be dem.*

Draw LI, LH, LK; then HK × LM *f* is = 2 triang. LHK. therefore 30 HK × LM *g* = 60 HLK = 20 HIK *h* = to the ſuperficies of an Icoſaedron. *Which was to be dem.*

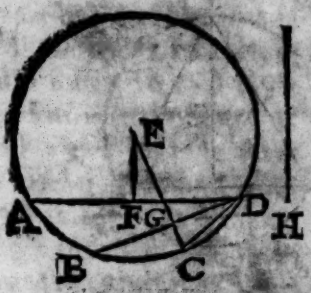
a 8. 1.
b 4. 1.
c 15. 5.
d 6. *ax.*
e 17. 3.
f 41. 1.
g 15. 5.
h 16. 13.

Coroll.

CD × FG. HK × LM *k* :: the ſuperficies of a dodecaedron to the ſuperficies of an Icoſaedron.

k 15. 5.

PROP. V.



The ſuperficies of a Dodecaedron has to the ſuperficies of an Icoſaedron inſcribed in the ſame ſphere, the ſame proportion that H the ſide of a cube has to AD the ſide of an Icoſaedron.

Let the circle ABCD *a* incloſe both the pentagone of a dodecaedron, and the triangle of an Icoſaedron; whoſe ſides are BD, AD. upon which from the center E let fall the perpendiculars EF, EGC, and draw CD.

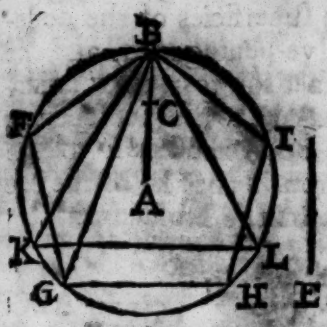
a 3. 13.

Becauſe EC + CD. EC *b* :: EC. CD. thence EG ($c \frac{1}{2}$ EC + CD.) EF ($d \frac{1}{2}$ EC) *e* :: EF. EG — EF. ($\frac{1}{2}$ CD.) but H. BD *f* :: BD. H — BD, *g* therefore H. BD :: EG. EF. conſequently H × EF = BD × EG. wherefore ſince H. AD *h* :: H × EF. AD × EF. *k* thence ſhall be H. AD :: BD × EG. AD × EF *l* :: the ſuperficies of a dodecaedron to the ſuperficies of an Icoſaedron. *Which was to be dem.*

b 9. 13.
c 1. 14.
d *cor.* 12. 13.
e 15. 5.
f *cor.* 17. 13.
g 2. 14.
h 1. 6.
k 7. 5.
l *cor.* 4. 14.

PROP.

PROP. VI.



If a right line AB be cut in extreme and mean proportion, then as the right line BF, containing in power that which is made of the whole line AB, and that which is made of the greater segment AC, is to the right line E containing in power that which is made of the whole line AB, and that which is made of the leſſer ſegment BC; ſo is the ſide of the cube BG to the ſide of an Icoſaedron BK inſcribed in the ſame ſphere with the cube.

In the circle, whoſe ſemidiameter is AB, inſcribe BFGHI the pentagone of a dodecaedron, and BKL the triangle of an Icoſaedron, *a* wherefore BG ſhall be the ſide of a cube inſcribed in the ſame ſphere. therefore BKq *b* = 3 ABq; and Eq *c* = 3 ACq. therefore BKq. Eq *d* :: ABq. ACq *e* :: BGq. BFq. wherefore by inverſion BGq. BKq :: BFq. Eq. *f* whence BG. BK :: BF. E. *Which was to be dem.*

a *cor.* 17. 13.
b 12. 13.
c 4. 13.
d 15. 5.
e 2. 14.
f 22. 6.

PROP. VII.

A Dodecaedron is to an Icoſaedron, as the ſide of a Cube is to the ſide of an Icoſaedron, inſcribed in one and the ſame ſphere.

Becauſe *a* the ſame circle comprehends both the pentagone of a dodecaedron, and the triangle of an Icoſaedron, *b* the perpendiculars drawn from the center of the ſphere to the planes of the pentagone and triangle, ſhall be equal one to another. Therefore if the Dodecaedron and Icoſaedron be conceived divided into pyramides, right lines being drawn from the center of the

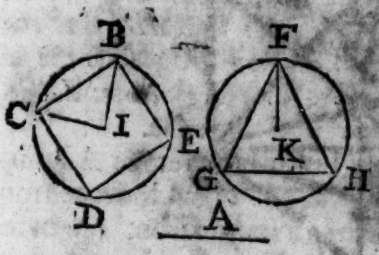
a 3. 14.
b 47. 1.

ſphere to all the angles, the altitudes of all the pyramides ſhall be equal one to the other.

c 5. *and* 12. Wherefore ſince the pyramides of equal heighth with the baſes, and the ſuperficies of the dodecaedron is equal to twelve pentagones, and the ſuperficicies of the Icoſaedron to twenty triangles, the dodecaedron ſhall be to the Icoſaedron, as the ſuperficies of the dodecaedron is to the

d 5. 14. ſuperficies of the Icoſaedron, *d* that is, as the ſide of the cube is to the ſide of the Icoſaedron.

PROP. VIII.



The ſame circle BCDE comprehends both the ſquare of the cube BCDE; and the triangle of the octaedron FGH inſcribed in one and the ſame ſphere.

Let A be the diameter of the ſphere. Becauſe

a 15. 13. Aq *a* = 3 BCq *b* = 6 BIq; and alſo Aq *c* = 2

b 47. 1. GFq *d* = 6 KFq; thence ſhall BI be = KF. *e*

c 14. 13. therefore the circle CBED = GFH. *Which*

d 22. 13. *was to be demonſtrated.*

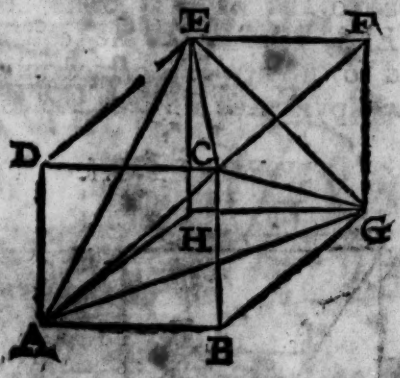
e 2. *def.* 1.

The End of the fourteenth Book.

THE

THE FIFTEENTH BOOK OF EUCLIDE's ELEMENTS.

PROP. I.



IN a cube given ABGHDCFE to describe a pyramide AGEC.

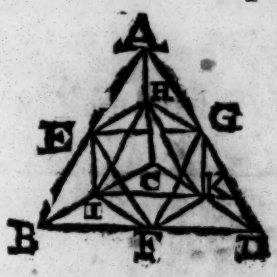
From the angle C draw the diameters CA, CG, CE; and connect them with the diameters AG, GE, EA. All which are *a* equal among themselves, as being the diameters of equal squares: therefore the triangles CAG, CGE, CEA, EAG are equilateral and equal; and consequently AGEC is a pyramide, which insists upon the angles of the cube, and therefore *b* is inscribed in it. *Which was to be done.*

a 47. 1.

b 31. *def.* 11.

PROP.

PROP. II.



In a pyramide given ABD to deſcribe an octaedron EG KIFH.

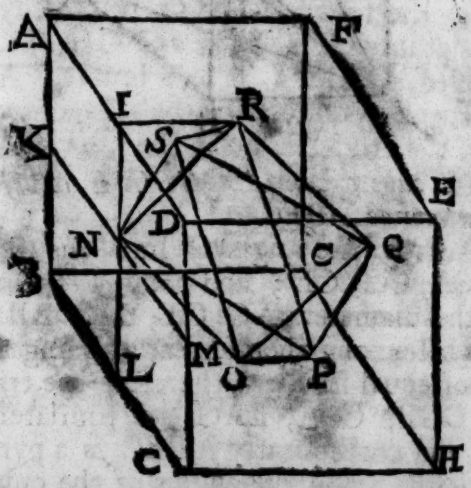
a 10. 1. *a* Biſect the ſides of the pyramide in the points E, I, F, K, G, H, which join with the right lines EF, FG, GE, *&c.* All theſe are

b 4. 11. *b* equal one to the other; conſequently the 8 triangles EHI, IHK, *&c.* are equilateral and

c 27. *def.* 15. equal, and ſo make *c* an octaedron deſcribed

d 31. *def.* 11. in the given pyramide. *Which was to be done.*

PROP. III.



In a cube given CHGBDEFA to deſcribe an octaedron NPQSOR.

* 8. 4. Connect * the centers of the ſquares N, P, Q, S, O, R, with the twelve right lines NP, PQ, QS, *&c.*

a 4. 1. which are *a* equal among themſelves; and ſo

b 31. *and* 27. *def.* 11. make eight equilateral and equal triangles: wherefore *b* the Octaedron NPQSOR *b* is inſcribed in the cube. *Which was to be done.*

PROP.

PROP. IV.

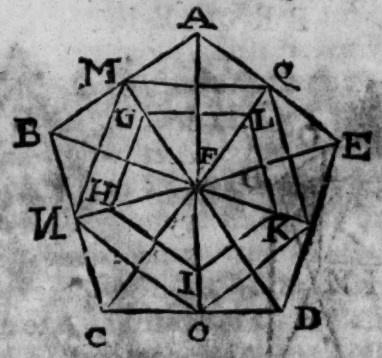
In Octaedron given ABCDEF, to inſcribe a cube.

Let the ſides of the pyramide E A B C D, whoſe baſe is the ſquare ABCD, be equally biſected by the right lines LM, MN, NO, OL, which are *a* equal and *b* parallel to the ſides of the ſquare ABCD. *c* then the quadrilateral LMNO is a ſquare. In like manner, if the ſides of the ſquare LMNO be equally biſected in the points G, H, K, I, and GH, HK, KI, IG connected, GHKI ſhall be a ſquare. And if in the other 5 pyramides of the octaedron, the centers of the triangles be in the ſame ſort conjoined with right lines, then other ſquares will be deſcribed like and equal to the ſquare GHKI. wherefore ſix ſuch ſquares ſhall make a cube, which ſhall be deſcribed within an octaedron, *d* being its eight angles touch the eight baſes of the octaedron in their centers. *Which was to be done.*

a 4. 1.
b 2. 6.
c 29. *def.* 1.
d 31. *def.* 11.

PROP.

PROP. V.



In an Icoſaedron given to inſcribe a Dodecaedron.

Let ABCDEF be a pyramide of the Icoſaedron, whoſe baſe is the pentagone ABCDE; and the centers of the triangles G,H,I,K,L; which connect with the right lines GH, HI, IK, KL, LG. Then GHIKL ſhall be a pentagone of the dodecaedron to be inſcribed.

For the right lines, FM, FN, FO, FP, FQ, paſſing by the centers of the triangles, *a* do equally divide their baſes into two parts. *b* therefore the right lines MN, NO, OP, PQ, QM *c* are equal one to the other; *d* whence alſo the angles MFN, NFO, OFP, PFQ, QFM are equal. therefore the pentagone GHIKL is equiangular. *e* and conſequently equilateral, being FG, FH, FI, FK, FL *f* are equal. And if in the other eleven pyramides of the Icoſaedron, the centers of the triangles be in like ſort conjoined with right lines, then will pentagones, equal and like to the pentagone GHIKL, be deſcribed. Wherefore 12 of ſuch pentagones ſhall conſtitute a dodecaedron; which alſo ſhall be deſcribed in the Icoſaedron, ſeeing the twenty angles of the dodecaedron conſiſt upon the centers of the twenty baſes of the Icoſaedron. Whereby it appears that we have deſcribed a dodecaedron in an Icoſaedron given. *Which was to be done.*

a *cor.* 3. 3.
b 4. 1.
c 4. 1.
d 8. 1.
e 4. 1.
f 12. 13.

EUCLIDE's

EUCLIDE's DATA.

A

Commentary or Preface written by the Philoſopher MARINUS, *on* EUCLIDE's *DATA.*

IN the firſt place we ought to ſet down (as a Foundation) what that is, which we call *DATUM* or GIVEN; then to conſider the Profit and Utility thereof; and in the third place, to what Art or Science this Tract doth appertain.

The Word *DATUM* therefore is diverſly defined, for the Antients have defined it after one manner, and later Writers after another, whence it follows that it ſeemeth a difficult thing to give a true Explication thereof; for ſome of them have not delivered the Definition of the Word; but have with much Labour and Trouble ſought certain Proprieties thereof, and ſome others collecting and mingling what hath been delivered

delivered by others before, have endeavored to define the Word *DATUM*; but not ſo exquiſitely but that they have contradicted themſelves; altho' what hath been ſaid by all of them, ſeems to be grounded on one and the ſame notion and ſuppoſition; for they all take the Word *DATUM* to be a thing compriſed; and therefore among ſuch as have endeavored to deſcribe it moſt ſimply, and with ſome ſimple difference, ſome of them have taken the Word *DATUM* to be the ſame with *ORDINATUM*, and ſo *Apollonius* underſtands it in his Tract of Inclinations, and in his univerſal Tract; and ſo others, as *Diodorus* takes it to be *COGNITUM* KNOWN; for in this Signification he takes the right line and the angles to be given, and all that may arrive to our Knowledge, altho' we may not be able well to expreſs it. But others have believed that it hath the ſame ſignification as the Word [*Effabile*] that may be declared, and ſo *Ptolomy* would have it, who calls thoſe things GIVEN, whoſe meaſure is known whether preciſely, or near the matter. Others alſo have thought the Word *DATUM* to be what is granted us by the Propoſer in the Hypotheſis; being that in the firſt Elements, a point given, and a right line given, is diverſly taken (that is to ſay, that who ſo would give and determine the quantity of a right line) all which things ſignify ſome COMPREHENSION; and therefore of all theſe Definitions, thoſe are moſt agreeable, which do moſt openly declare the COMPREHENSION, as we ſhall make evident by what follows.

Let us now unfold the diverſe Opinions of thoſe, who writing the nature of *DATUM* GIVEN, have not taken one ſimple Mark, or only Character for its Definition; and let us reduce it as in a Summary or Epitomy, to the

end

end we may with the more eaſe know or number all their Differences. Some of them then have defined *DATUM* to be *Ordinatum* and *Porimon* together, and others *Ordinatum* and *Cognitum* together, and others *Porimon* and *Cognitum* together. Wherefore all ſeem to have ſo defined it as to have had regard to the *Comprehenſion*, or *Aſſuming* and *Invention* of the thing given; and to the end that we may the better conceive their Opinions, and that from the ſaying of many we may be able to draw a true Definition of what is propoſed, we will take notice in the firſt place of the Signification of all the ſimple Terms which they make uſe of, as alſo of the Terms oppoſed to them, to wit, *Inordinatum* and *Incognitum*, *Aporon* and *Irrational*; for thoſe things appertain to this Geometrical Buſineſs, to natural things, and to Mathematical Diſcipline.

Now we may call that *Ordinatum* (or Regulated) which doth always keep and obſerve that for which it is ſaid to be ordered, whether you regard its Magnitude or Species, or touching ſome other ſuch like thing: It is alſo thus defined, *Ordinatum* is that which cannot be done in divers manners, but in one only manner, and in ſome determined place: As for example, A right line drawn by two given points, is ſaid to be ordered, by reaſon it cannot be otherwiſe done, nor in divers manners. But an angle paſſing by two points is ſaid to be *Inordinatum* (or diſordinate and irregular) for that it is made in infinite and divers manners by a great or ſmall circle deſcribed by two points *ad infinitum*. Contrariwiſe, an angle conſtituted by three points, is ſaid to be *Ordinatum*, as alſo thoſe things which follow are ſaid to be *Ordinatum*, as to conſtitute an equilateral triangle on a right line, for it cannot be diverſly made, but unchangeably, on both the

given

extremities of the line. Again, To divide a given right line according to a given proportion, for that cannot be done but in one certain point. The things *Inordinatum* are such as are done contrary to those last mentioned, as to constitute a Scalene triangle, and to divide a right line indefinitely. Wherefore the Problem is ordered, is proposed in the determination, considering that a certain thing may be in one manner said to be *Ordinatum*, and *Inordinatum* in another, as an equilateral triangle, considering the equality of the sides, it is *Ordinatum*, but considering its Magnitude it is *Inordinatum*, being in no wise determined.

But we call that *Cognitum* which is notorious, as clear and comprehended of us, and *Incognitum* that which is not known, or comprehended of us, as the length of a way is said to be known, when we know how many Miles it contains; also that the three lines of a Rectiline triangle are equal to two right angles; and in like manner that the Binomial is Irrational, such things are known, as also that it is only one right line that can touch a spiral line from a given point without it, from both parts; for if there were yet another line, two right lines would enclose a space, which is impossible. Again, irrational things are not said to be unknown, but such of them only which are neither known nor comprehended of us.

Porimon is that which we may make and constitute, that is to say, bring to our Understanding. Again, it is defined thus, *Porimon* is that which may be exhibited by Demonstration, or which is apparent without Demonstration, as to describe a circle from a center and with a space, as also to constitute not only an equilateral triangle, but also a Scalene; or to find a *Binomium*, or to find two right lines rational, commensurable in power only, and other things

things which are known infinitely, are *Porimon*; as to describe a circle by two points.

Aporon is wholly opposite to *Porimon*, as for example, the Quadrature of a circle, for that it hath not as yet been found; altho' it be certainly known that it may be: Nevertheless the manner of finding it out hath not been to this present comprehended. But we speak here of that which is already known, which is called *Porimon principale*; for what hath not been as yet made, and yet nevertheless is possible, is called *Poriston*, (or feasible) altho' the Construction be yet unknown. But *Aporon*, as hath been afore said, is opposite to *Porimon*, and is that whose Nature is not as yet decided, nor well determined.

Effabile, that is to say, rational (or speakable and explicable) is that whose Magnitude, Species, and Position, we may be able to declare; but this Definition is a little too general, for properly, and according to it self, *Effabile* is that which is known by certain things, and according to a Measure given by Position, as of a span, or a finger's breadth, *&c.*

These things then being thus unfolded, we may easily perceive in what all those things that we have afore spoken of do agree together, and wherein they do differ; and first of all how *Ordinatum* and *Cognitum* do agree together, and likewise their opposites the one to the other, for it cannot be said that any one of those things by counterchanging is the other, nor yet that the one hath not more extent than the other, altho' they agree in many things, as to describe a right line by two points, and to constitute an equilateral triangle by three circles. But to square a circle, that is indeed *Ordinatum*, yet nevertheless, *Incognitum*. Also that at a point of a spiral line there is but one touch line, that is, of the

kind we call *Ordinatum*, and cannot be otherwiſe done, yet nevertheleſs the Demonſtration and Conſtruction thereof is not yet known. Again, the indefinite ſection, and the Conſtruction of the Scalenum is *Cognitum*; but is not *Ordinatum*; inſomuch as it is manifeſt, that amongſt thoſe things which are *Ordinatum*, there are ſome that are *Cognitum*, and others that are *Incognitum*; and contrariwiſe, that amongſt the things that are *Cognitum*, there are ſome that are *Ordinatum*, and others that are *Inordinatum*; and therefore thoſe things anſwer one another, as among Living Creatures, that which hath reaſon, with that which hath Feet, for there is no equality amongſt them, neither doth the one extend more than the other.

In like manner, *Ordinatum* and *Inordinatum* agree together, reſpecting *Porimon* and *Aporon*; ſeeing that between them there is a very great Reſemblance, and becauſe that they do differ only in the manner before expreſſed; for in truth the ſpiral line is *Ordinatum*; but it was not *Porimon* before *Archimedes*; and by the ſame reaſon thoſe things that are inordinate and known by an infinity of ways and means are *Porimon*, if any one ſhall undertake to invent their Conſtitution and Conſtruction. Yet nevertheleſs they are not ordinate, as to conſtitute a Scalenum triangle, it being no difficult thing to make known the Conſtitution thereof by an equilateral triangle, yea it is moſt eaſy, altho' it be inordinate and known by an infinity of ways.

And in the ſame manner do agree *Ordinatum* and *Inordinatum*, together with *Effabile* and *Irrationale*; for they agree together in many things, differing nevertheleſs by the foregoing reaſon, ſeeing thoſe things there mentioned bear no equality to each other, neither doth one

thing

thing contain the other; for all Binomiums and ſuch as are taken as Irrationals, are indeed ordinate, but yet they are not *Effabile*, or expreſſible, or to be unfolded, as the diameter of the ſquare is in reſpect of its ſide. Now touching *Effabile*, there are divers inordinate; becauſe they are diverſly known, and indeterminately, for a Scalenum triangle may be meaſured by a defined and propoſed meaſure, as explicable, altho' it be inordinate.

Now it is eaſy to ſee the agreement that there is between *Cognitum* and *Porimon*, but it is a difficult thing to expreſs or unfold their difference; foraſmuch as in their Natures they come ſo near to one another, as that there ſeems to be an equality between them: Nevertheleſs, there will ſome difference appear to him that ſhall conſider it more ſtrictly; for let it be conſented to that there can be only one line that can touch a ſpiral line in a certain point, that is *Cognitum*, yet notwithſtanding the Problem is not *Porimon*, it being not as yet comprehended; ſo as that all that which is *Cognitum*, is not therefore *Porimon*. But all that which is *Porimon* is alſo *Cognitum*, and therefore *Cognitum* appears to be of a greater extent than *Porimon*.

Now *Cognitum*, *Porimon*, and *Effabile*, do agree in ſome certain things, and do differ in other things by the ſame reaſon before alledged; for thoſe lines which are there called Irrationals are in truth known; and yet nevertheleſs are not *Effabile* or explicable. Contrariwiſe, every number is indeed *Effabile*, and yet every number is not *Cognitum*. But *Effabile* is always of its own nature expreſſible, altho' that ſome lengths may be now *Effabile*, and at another time not, if it be examined with ſome other according to one and the ſame meaſure. But alſo that ſame length is ſometimes known,

and other times not, tho' they wholly agree with one another. Now it is a difficult thing to find ſomething that ſhall be *Effabile* and *Incognitum*, for *Cognitum* ſeems to be of a greater extent than *Effabile*, and by thoſe things it is manifeſt that *Porimon* and *Aporon* do differ from RATIONAL or *Effabile*, and from IRRATIONAL, for of IRRATIONAL ſome of them may be *Porimon*; but of RATIONAL none of them can be Irrationals; and therefore it is very eaſy to perceive in what the before expreſſed things agree. Notwithſtanding they ſeem to agree together, in ſuch ſort as that *Porimon* ſeems to be of a greater extent than *Effabile*.

Now by theſe things we may come to know the difference of thoſe things that have been before ſpoken of, for in truth *Effabile* and *Irrationale* are ſo termed in reſpect of meaſure, which notwithſtanding is not as yet arrived to our Underſtanding, ſeeing that ſomething that is rational, may be as yet unknown to us, and in like manner may be rational, and yet may never be comprehended ſo to be. But *Ordinatum* and *Inordinatum* is ſo termed according to it ſelf, and according to the proper nature of the thing on which we contemplate, altho' it be not comprehended by us. As *Archimedes* had perceived ſome things to be ordinate from the nature of the things, the which *Serenus* had before contemplated. But *Cognitum* and *Incognitum* is ſpoken in reſpect of us, ſo as the things before mentioned do differ among themſelves; for theſe have reſpect to us, the others, ſome of them to their proper nature, and the reſt to meaſure.

Having then explained the agreements and differences of the things that have been propoſed, it remains now that we conſider what is meant by the Word *DATUM*, for of all thoſe

thoſe that believe the Word *DATUM* to be that which is conſented to by the Propoſer in the Hypotheſis, are wide from what is ſought; becauſe that all the Elements of the things GIVEN are not compoſed of this ſort of GIVEN, which is according to the Hypotheſis, as may be ſeen in thoſe Tracts which have been made on this Subject GIVEN. Wherefore waving this Opinion, let us judge of the Definitions of others.

Then, that which is conſented to, or granted in the Hypotheſis, is ſomething which is conſequently known by the Principles; but ſuch as make uſe of Definitions of one only Word, do define it and remark it by ſome one of the before mentioned, as hath been ſpoken in the beginning, ſo as that almoſt all ſeem to have had this common notion of GIVEN, to wit, that it is comprehended even as the Word *DATUM*, doth alſo manifeſt it to be; and amongſt thoſe, theſe are the chief that do define it by the Hypotheſis or Suppoſition; and others have had regard to what is conſented to or granted. But we making uſe of the ſaid things as of a Rule and Direction to judge rightly, we may be able to find out a perfect Definition of *DATUM*; for it is certain that it ought to equal and be convertible with the thing defined, which is one thing proper to good Definitions. Now ſuch ſeems to be the Definition of the thing propoſed, which among the moſt ſimple and plain Expoſitors is defined *Porimon*, and amongſt the more acute, that which defineth it to be *Porimon* and *Cognitum* together; but all the reſt are imperfect; for that which defineth it *Ordinatum* is not ſufficient for the Comprehenſion and Knowledge of *DATUM*; becauſe that neither wholly ordinate nor alone ordinate, is not compriſed, ſeeing that there are things inordinate that have

the ſame Condition, as hath been ſhewn. Again, that reaſon gives not Satisfaction neither, which deſcribes it to be *Cognitum*; for all that is known is not comprehended, altho' that alone *Cognitum* be comprehended. Moreover, that alſo is not perfect which defineth it to be *Effabile*, for *Effabile* is not alone comprehended; ſeeing that ſome of the Irrationals are alſo comprehended. In like manner, all *Effabile* is not comprehended, as hath been before declared. Now amongſt the Definitions which expound it by the only Word, there remains that which defineth it to be *Porimon*, which ſeemeth greatly to manifeſt the Comprehenſion; for whole *Porimon* and alone *Porimon* is comprehended. Wherefore *EUCLIDE* himſelf uſeth ſuch a Definition in a Deſcription of all the kinds of GIVEN by him conceived and regarded. But amongſt ſuch Definitions as are compounded, that is a perfect Definition which defineth *DATUM* to be *Cognitum* and *Porimon* together, having *Cognitum* for analogical kind, and *Porimon* for difference; but that is imperfect which hath *Ordinatum* and *Porimon* together; for thoſe things which are ſuch, are not alone GIVEN, and that which defineth it *Ordinatum* and *Effabile* together, comprehendeth likewiſe the GIVEN, with the defect or want. But that of *Cognitum* and *Ordinatum* together, is not to be received or admitted, becauſe it doth exceed what is defined, for ſuch is not given alone: Therefore thoſe only which have declared that *DATUM* is *Cognitum* and *Porimon* together, ſeem to have attained the notion of GIVEN, for that which is ſuch is all, and alone comprehended, which two things ought to be in thoſe Definitions that are well given. But the former comes near to thoſe which have thus defined it: *DATUM* is that to which we may find an equal, according to thoſe things

we

we have propoſed in the firſt Principles and Hypotheſis, of which number *EUCLIDE* is one, making uſe throughout of the Word πορισάσθα, which ſignifies to exhibit or invent, altho' he leaves *Cognitum* as a Conſequent of *Porimon*; ſome one nevertheleſs might reprove him, for that in the firſt place he hath not defined *DATUM* in general; but immediately ſome of the kinds of GIVEN, altho' in his Elements of GEOMETRY, he hath defined the line ſimple before the Species or Kinds.

What is the Utility and Profit that ariſeth from this Tract of DATA, *or things GIVEN.*

AFter having explained univerſally, and according to what ſeemed neceſſary for our preſent Uſe, what this Word *DATUM* ſignifieth: It follows to ſhew the Utility of this Tract. Now this Tract is ſuch, as that it is not only ordained and inſtituted for its own reſpect, but for ſome other thing; for it is very neceſſary to a place which is called Reſolved, and we have already declared elſewhere how much Strength a reſolved place doth obtain in Mathematical Diſciplines, as alſo in Opticks and Cannons, which come very near to them, as well for that Reſolve is an Invention of the Demonſtration, as for that in ſuch like things it ſerves us much for the Invention of the Demonſtration, or for that it is much more excellent to meet with a Reſolutive Power, than to enjoy divers particular Demonſtrations.

To what Art or Science this Tract is referred.

NOW ſeeing the Conſideration of GIVEN is uſeful and profitable in all theſe kinds

of Arts, for that it ſerveth much to RESOLUTION, it may well be ſaid to be recalled, not only to one only Science, but to the Mathematicks univerſally which treat of Numbers, Time, Swiftneſs, and ſuch like things, which treateth likewiſe of Reaſons, as alſo of Proportions, and in a word of all *Medietez*: Wherefore for the perfect and demonſtrative Knowledge of things GIVEN, being of ſo great Utility, *EUCLIDE* hath taken pains to frame this Book of things GIVEN, which Author amongſt all ſuch as have compoſed the Elements of Geometry, hath juſtly deſerved the firſt place and rank, and who having invented the Elements, or rather the Introductions almoſt of all Mathematical Diſciplines, to wit, of all Geometry in 13 Books, of Aſtronomy, of Muſick, and Opticks, he hath left in Writing the Elements RESOLUTIVE, in this Treatiſe of things GIVEN; but as he was a Geometrician, he hath particularly accommodated to Magnitudes, that was of the GIVEN; yet nevertheleſs common in other things, which Method hath alſo been obſerved by him, when treating univerſally of Reaſons and Proportions, he appropriates them to the Magnitudes mentioned in his Fifth Book of Planes.

Now it hath been declared in general what is the meaning of *DATUM*, to what Science it appertaineth, and how profitable the Contemplation thereof is. We will add to what hath been ſaid, the Deſcription of this Science which treats of things GIVEN; ſeeing that it is (as appears by what hath been ſaid) a Comprehenſion in all manners of things GIVEN; and of their Accidents and Proprieties. But having reſpect to the propoſed Book, we ſhall declare it to be an Elementary Doctrine of the whole Knowledge of things GIVEN, whence it follows that it will be very

very profitable, as also the things therein contained, seeing they refer to things GIVEN.

Now this Book is divided according to the Species or Kinds of the things GIVEN, and in the first Section are contained those things which are given by Reason. Secondly, such as are given by Position: and lastly, such as are given by Species or Kind; for that which is given by Magnitude is simple, and particularly contained in the others, and principally in those things given by Species or Kind. Now he hath begun with those things given by Reason and Position, forasmuch as those that are given by Species are constituted of them. *EUCLIDE* gives yet another Division to this Book, for that he divides it into universal Magnitudes, Lines, and Superficies, and into circular Theoremes, which Order he hath also observed in the Definitions and Suppositions of this Book. Moreover, he useth a certain way of instructing, which proceeds not by Composition, but by Resolution, as *Pappus* hath amply set down in his Commentaries on this Book.

The End of the PREFACE of MARINUS.

DEFI-

DEFINITIONS.

I. PLanes or Spaces, Lines, and Angles, to which we may find others equal, are ſaid to be given by Magnitude.

II. *A Reaſon is ſaid to be given, when we may find one of the ſame or equal thereto.*

III. *Rectiline figures, whoſe angles are given, and alſo the reaſon of the ſides to one another, are ſaid to be given by Species or Kind.*

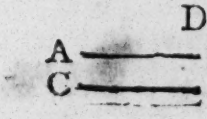
IV. *Points, Lines, and Angles, which have and keep always one and the ſame place and ſituation, are ſaid to be given by Poſition or Situation.*

V. *A Circle is ſaid to be given by Magnitude, when the ſemidiameter thereof is given by Magnitude.*

VI. *A Circle is ſaid to be given by Poſition, and by Magnitude when the center thereof is given by Poſition, and the ſemidiameter by Magnitude.*

VII. *Segments of Circles, whoſe angles and baſes are given by Magnitude, are ſaid to be given by Magnitude.*

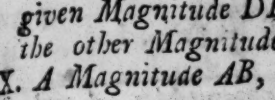
VIII. *Segments of a Circle, whoſe angles are given by Magnitude, and the baſes of the ſegments by Poſition and Magnitude, are ſaid to be given by Poſition and by Magnitude.*



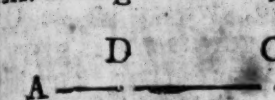
IX. *A Magnitude AB, is greater than another Magnitude C, by a given Magnitude BD, when having taken away the given*

given Magnitude DB, the reſt AD, is equal to the other Magnitude C.

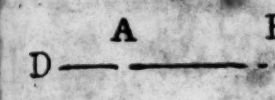
X. *A Magnitude AB, is leſs than another Magnitude C, by a given Magnitude BD. when having added thereto the given Magnitude BD, the whole AD is equal to the other magnitude C.*



XI. *A Magnitude AB, is ſaid to be greater than another magnitude CB, by a given magnitude AD, and in reaſon, when taking from the ſame magnitude the given magnitude AD, the reſt DB, hath to the other magnitude CB, a given reaſon.*



XII. *A magnitude AB is ſaid to be leſs than another magnitude BC, by a given magnitude AD, and in reaſon, when the given magnitude AD being added thereto, the whole DB hath to the other magnitude BC, a given reaſon.*



XIII. *A right line is ſaid to be drawn down from a given point, unto a right line given in Poſition, the right line being drawn in a given angle.*

XIV. *A right line is ſaid to be drawn up from a given point, to a right line given in Poſition, the right line being drawn in a given angle.*

XV. *A right line is againſt another right line in Poſition, when it is drawn parallel thereto through a given point.*

PROPO-

PROPOSITION I.

TWO magnitudes A and B, being given, the reaſon they have to one another A to B, is alſo given.

Demonſtration. For ſeeing that the magnitude A is given, [a] we may find one equal thereto, which let be C. Again, foraſmuch as the magnitude B is given, we may alſo find one equal to that, and let that be D. Therefore ſeeing that A is equal to C, and B to D, as A is to C, [b] ſo is B to D, and by permutation, [c] as A ſhall be to B, ſo C ſhall be to D. Therefore [d] the reaſon of A to B is given, for it is the ſame reaſon as of C to D, as we have found, and which ought to be demonſtrated.

a 1. *def.*
b 7. 5.
c 16. 5.
d 2. *def.*

PROP. II.

If a given magnitude A, hath to ſome other magnitude B, a given reaſon, that other magnitude B, is alſo given by magnitude.

Demonſtr. For ſeeing that A is given, we may find one equal thereto, which let be C: And foraſmuch as the reaſon of A to B, is alſo given, we may find [a] one of the ſame. Let it be found, and let the reaſon be of C to D. Now ſeeing that as A is to B, ſo C is to D; and by permutation, as A is to C, ſo B is to D: But A is equal to C, therefore [b] B ſhall be alſo equal to D. Therefore [c] the magnitude B is given, ſeeing that thereto there hath been found one equal, to wit, D.

a 2. *def.*
b 14. 5.
c 1. *def.*

PROP.

PROP. III.

If given magnitudes AB and BC, are compounded, that magnitude AC, that is compounded of them ſhall be alſo given.

Demonſtr. For ſeeing that AB is given, we may find one equal to it, which let be DE. Again, ſeeing that BC is given we may alſo find one equal to that, which let be EF. Wherefore ſeeing that DE is equal to AB, and EF is equal to BC, the whole AC *a* is equal to the whole DF. Therefore AC is given, ſeeing that DF is propoſed equal thereto.

a 2. *ax.* 1.

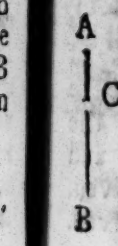
PROP. IV.

If from a given magnitude AB, there be taken away a given magnitude AC, the remaining magnitude CB is alſo given.

Demonſtr. Foraſmuch as AB is given, we may find one equal thereto, which let be DE. Again, ſeeing that AC is given, we may alſo find one equal to it, which let be DF. Seeing then that the magnitude AB is equal to the magnitude DE, and the magnitude AC to the magnitude DF; the reſt CB *a* ſhall be equal to the reſt FE. Wherefore CB is given, for to it there hath been found an equal, to wit, FE.

a 3. *ax.* 1.

PROP. V.



If a magnitude AB, hath a given Reaſon to ſome part thereof AC, it will have alſo a given reaſon to the part remaining CB.

Demonſtr. Let DE be expoſed as a given magnitude, and ſeeing that the reaſon of the magnitude AB,

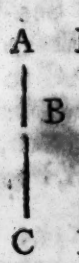
to

a 2. *def.* to the magnitude AC, is given, *a* we may find one of the ſame, which let be DE to DF; therefore the reaſon of the ſame DE to DF is
b 2. *prop.* given. But DE being given, ſo is *b* alſo its
c 4. *prop.* part DF; and conſequently, *c* the reſt FE:
d 1. *prop.* Therefore *d* ſeeing that DE and FE are given, the reaſon of the ſame DE to FE, is alſo given. And foraſmuch as DE is to DF, as AB is to AC, and by converſion, as DE to FE,
19. 5. ſo AB is to CB. But the reaſon of DE to FE is given, as hath been demonſtrated; therefore the reaſon of AB to CB is alſo given.

Scholium.

From this it is evident that if a magnitude hath to ſome part thereof a given reaſon, by diviſion, the reaſon that one part hath to the other, ſhall be alſo given. For ſeeing that as DE is to FE, ſo is AB to CB; by diviſion, as DF to FE, ſo AC to CB. But it hath been demonſtrated that the parts DF and FE are given, and conſequently their reaſon is alſo given: In like manner, therefore the reaſon of AC to CB is given.

PROP. VI.



If two magnitudes AB and BC, having to one another a given reaſon, are compounded, the magnitude AC compounded of them, ſhall allo have a given reaſon to each of them AB and BC.

Demonſtr. Let the given magnitude DE be expoſed, and ſeeing that the reaſon of AB to BC is given; let there be made one and the ſame of the ſaid DE to EF; therefore the reaſon of the
a 2. *prop.* ſame DE to EF is given; and therefore *a* the magnitude DE being given, both the one and the other of them DE and FE, is given.
b 3. *prop.* Wherefore *b* the whole DF ſhall be alſo given.

There-

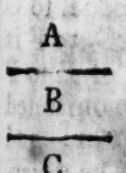
Therefore *c* the reaſon of the ſame DF to each of them DE and EF, ſhall be given. And foraſmuch then as AB is to BC, ſo is DE to EF; in compounding, *d* as AC is to BC, ſo is DF to EF: Therefore by converſion, as AC to AB, ſo is DF to DE. Therefore as the whole DF is to each of the other magnitudes DE and EF, ſo the whole AC is to each of the magnitudes AB and BC: Therefore *e* the reaſon of the ſame AC to each of the magnitudes AB and BC is given. c 1. *prop.* d 18. 5. e 2. *def.*

PROP. VII.

If a given magnitude AB, be divided according to a given reaſon AC to CB, each ſegment AC and CB is given.

Demonſtr. For ſeeing the reaſon of AC to CB is given, the reaſon of *a* AB to each of them (AC and CB) is alſo given. But AB is given: Therefore *b* each of the ſegments AC and CB is alſo given. a 6. *prop.* b 2. *prop.*

PROP. VIII.



Magnitudes A and C, which have to one and the ſame a given reaſon B, ſhall be to one another in a given reaſon, A to C.

Demonſtr. For let the given magnitude D be expoſed, and ſeeing that the reaſon of A to B is given, let the ſame be done of the ſaid D to E. Now ſeeing that D is given, *a* E is alſo given. Again, ſeeing that the reaſon of B to C is given, let the ſame be done of E to F. But E is given, and therefore F is alſo given. But ſeeing that D is a 2. *prop.*

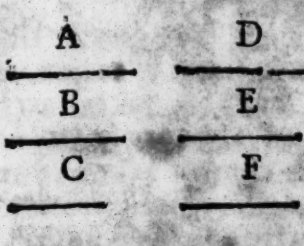
given,

given, *b* the reaſon of the ſame D to F is given; and ſeeing that as A to B, ſo D to E, and as B to C, ſo is E to F; in reaſon of equality, *c* as A is to C, ſo is D to F; but the reaſon of D to F is given. Therefore the reaſon of A to C is alſo given.

b 1. *prop.*

c 22. 5.

PROP. IX.



If two or more magnitudes A, B, and C, are to one another in a given reaſon, and that the ſame magnitudes A, B, and C, have to other magnitudes D, E, and F, given reaſons, altho' they be not the ſame, thoſe other magnitudes D, E, and F, ſhall be alſo to one another in given reaſons.

Demonſtr. Foraſmuch as the reaſon of A to B is given, as alſo that of A to D, the reaſon of D to B ſhall be given: But the reaſon of B to E is alſo given; therefore the reaſon of the ſame D to E ſhall be in like manner given. Again, ſeeing that the reaſon of B to C is given, and alſo that of B to E, the reaſon of E to C ſhall be given. But the reaſon of C to F is alſo given. Therefore *a* the reaſon of E to F ſhall be given. But it hath been demonſtrated that the reaſon of D to E is alſo given; and therefore *b* the reaſon of D to F ſhall be given. Therefore the magnitudes D, E, and F, are to one another in given reaſons.

a 8. *prop.*

b 8. *prop.*

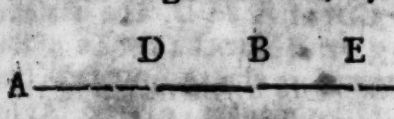
PROP. X.

If a magnitude AB, be greater than another magnitude BC, by a given magnitude, and in reaſon, the magnitude

nitude AC compounded of both, ſhall be alſo greater than that ſame magnitude, by a given magnitude, and in reaſon: But if that compounded magnitude be greater than the ſame magnitude, by a given magnitude, and in reaſon; either the remainder ſhall be alſo greater than that ſame by a given magnitude, and in reaſon; or elſe the ſame remainder is given with the following, to which the other magnitude hath a given reaſon.

Demonſtr. For ſeeing that AB is greater than BC by a given magnitude, and in reaſon, let the given magnitude AD be taken away. Therefore *a* the reaſon of the remainder DB to BC is given; and in compounding, *b* the reaſon of DC to BC is alſo given. But the magnitude AD is alſo given; therefore AC is greater than the ſame BC by a given magnitude, and in reaſon. a 11. *def.* b 6. *prop.*

Again, Let the magnitude AC be greater than the magnitude BC, by a given magnitude, and in reaſon:



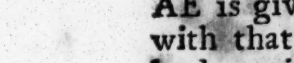
I ſay that the reſt AB, is either greater than the ſame BC by a given magnitude and in reaſon, or that the ſame AB, with that which followeth, to which BC hath a given reaſon, is given.

Foraſmuch as the magnitude AC is greater than the magnitude BC, by a given magnitude, and in reaſon, cut off from it the given magnitude: Now the ſame given magnitude is either leſs than the magnitude AB, or greater: Let it in the firſt place be leſs, and let it be AD. Therefore the reaſon of the remainder DC to CB is given. Wherefore by diviſion, the reaſon of DB to BC is given. But the magnitude AD is alſo given; therefore the magnitude AB is greater *c* than the magnitude BC by a given magnitude, and in reaſon. Now let the c 11. *def.*

given magnitude be greater than the magnitude AB, and let AE be put equal thereto; therefore *d* the reason of the remainder EC to CB is given; and by converſion, *e* the reaſon of the ſame BC to BE, is alſo given. But the ſame EB with BA is given, for that the whole AE is given: Therefore there is given AB, with that which follows BE, to which BC hath a given reaſon.

d 11. *def.*
e 5. *prop.*

PROP. XI.



If a magnitude AB, be greater than a magnitude BC, by a given magnitude, and in reaſon, the ſame magnitude AB, ſhall be alſo greater than the magnitude compounded of them by a given magnitude, and in reaſon, and if the ſame magnitude be greater than the two others together by a given magnitude, and in reaſon, that ſame magnitude ſhall be alſo greater than the reſt by a given magnitude, and in reaſon.

Demonſtr. For ſeeing that the magnitude AB is greater than BC by a given magnitude, and in reaſon; let there be taken from it a given magnitude AD: Therefore *a* the reaſon of the reſt DB to DC, is given, and therefore *b* the reaſon of DC to BD ſhall be alſo given: Let the ſame be done of AD to DE, therefore the reaſon of the ſame AD to DE is given. But AD is given, therefore *c* DE is alſo given, and conſequently, *d* the reſt AE, is alſo given. But ſeeing that as AD is to DE, ſo is DC to BD; by permutation, *e* as AD is to DC, ſo is DE to DB: Therefore by compounding, *f* as AC is to CD, ſo is EB to DB; and by permutation, *g* as AC is to EB, ſo is DC to DB. But the reaſon of DC to DB is given: Therefore alſo is AC to EB, and conſequently, that of EB to AC. But it hath been demonſtrated that AE is given, therefore *h* AB is greater than AC by a given magnitude, and in reaſon.

a 11. *def.*
b 6. *prop.*
c 2. *prop.*
d 4. *prop.*
e 16. 5.
f 18. 5.
g 16. 5.
h 11. *def.*

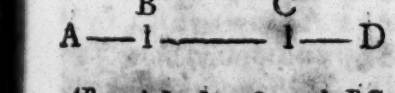
But

But now let AB be greater than AC by a given magnitude, and in reaſon: I ſay that the ſame AB is alſo greater than the reſt BC, by a given magnitude, and in reaſon.

For ſeeing that AB is greater than AC by a given magnitude, and in reaſon. Let the given magnitude AB be cut off there-from: Therefore *i* the reaſon of the remainder EB to AC is given, and conſequently, alſo ſhall be given that of AC to EB. Let the ſame be done of AD to DE, therefore the reaſon of AD to DE is given; and by converſion, *k* the reaſon of AD to AE ſhall be alſo given, and conſequently that of AE to AD. Now AE is given, therefore the whole AD *l* ſhall be alſo given; and ſeeing that as the whole AC is to the whole EB, ſo the part cut off AD, is to the part cut off ED, ſo alſo ſhall be *m* the remainder DC to the remainder DB. But the reaſon of AC to EB is given: Therefore alſo ſhall be given that of DC to DB. Wherefore by diviſion, *n* the reaſon of BC to DB is given; and conſequently alſo ſhall be given that of DB to BC. But it hath been demonſtrated that AD is given: Therefore *o* AB is greater than the ſame BC by a given magnitude, and in reaſon.

i 11. *def.*
k 5. *prop.*
l 2. *prop.*
m 19. 5.
n *ſchol.* 5. *prop.*
o 11. *def.*

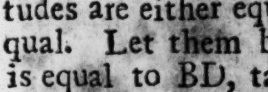
PROP. XII.



If there be three magnitudes AB, BC, and CD, and that the firſt AB, with the ſecond BC, to wit AC, be given. But the ſecond BC, with the third CD, to wit, BD, be alſo given: Either the firſt AB ſhall be equal to the third CD, or the one ſhall be greater than the other by a given magnitude.

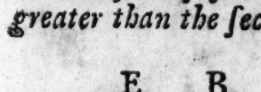
Demonſtr. Foraſmuch as each of the magnitudes AC and BD are given, the given magni-

tudes are either equal to one another, or unequal. Let them be firſt equal: Therefore AC is equal to BD, take away the common mag-
a 3. *ax.* 1. nitude BC, and there will remain *a* AB, equal to CD. But ſuppoſe them to be unequal, as in this ſecond figure, and let BD be greater than AC: Let then BE be put equal to AC. Now ſeeing that AC is given, BE is alſo given. But the whole BD is alſo
b 4. *prop.* given, the reſt ED *b* ſhall be ſo alſo; and foraſmuch as BE is equal to AC, taking away
c 3. *ax.* 1. the common magnitude *c* BC, there will remain AB equal to CE. But ED is given: Therefore CD is greater than AB by the given magnitude ED.



Scholium.

And if the firſt with the ſecond, to wit, AC, were greater than the ſecond with the third, to wit, BD, as in the other figure, CE would be made equal to the ſame BD, and by the ſame reaſons as was above demonſtrated, that AE is given and equal to CD; and therefore AB greater than CD by a given magnitude.



PROP. XIII.

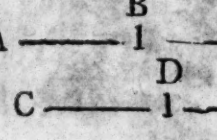
If there be three magnitudes AB, CD, and E, and that the firſt of them AB, hath a given reaſon to the ſecond CD; but the ſecond CD is greater than the third E, by a given magnitude, and in reaſon, alſo the firſt AB, ſhall be greater than

than the third E, by a given magnitude, and in reaſon.

Demonſtr. For ſeeing that CD is greater than E by a given magnitude, and in reaſon, let the given magnitude CF be taken there-from: Therefore the reaſon of the reſt FD to E is given. And foraſmuch as the reaſon of AB to CD is given, let the ſame be done of AH to CF. Therefore the reaſon of the ſame AH to CF is given. But CF is given: Therefore *a* AH is alſo given. And ſeeing that as the whole AB is to the whole CD, ſo the part cut off AH is to the part cut off CF, and ſo *b* alſo the reſt HB is to rhe reſt FD, the reaſon of the ſame HB to FD is alſo given. But the reaſon of FD to E is alſo given: Therefore *c* the reaſon of HB to E is given. But it hath been demonſtrated that AH is given: Therefore *d* AB is greater than the ſaid E by a given magnitude, and in reaſon.

a 2. *prop.*
b 19. 5.
c 8. *prop.*
d 11. *def.*

PROP. XIV.



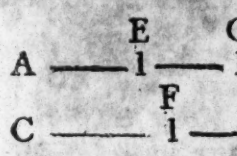
If two magnitudes AB and CD, have to one another a given reaſon, and that to each of them there be added a given magnitude, to wit, BE and DF; either the whole AE and CF ſhall have to one another a given reaſon, or the one ſhall be greater than the other by a given magnitude, and in reaſon.

Demonſtr. For ſeeing that each of thoſe magnitudes BE and DF, is given, *a* the reaſon of the ſaid BE and DF is alſo given; and if that reaſon be the ſame with that of AB to CD, that of the whole AE to the whole CF, *b* ſhall be the ſame; and therefore the reaſon of the ſaid AE to CF is given.

a 1. *prop.*
b 12. 5.

Now let the reaſon of BE to DF be not the ſame, with that of AB to CD, and let it be as AB to CD, ſo BG to DF. Therefore the reaſon of the ſaid BG to DF is given. But
c 2. *prop.* the magnitude DF is given, therefore *c* BG is alſo given; and ſeeing that the whole BE
d 4. *prop.* is given, *d* the reſt GE ſhall be alſo given. But foraſmuch as AB is to CD, as BG is to
e 12. 5. DF, *e* ſo alſo is the whole AG to the whole CF; and therefore the reaſon of the ſaid AG to CF is given: But the magnitude GE is
f 11. *def.* given: Therefore *f* the magnitude AE is greater than the magnitude CF by a given magnitude, and in reaſon.

PROP. XV.



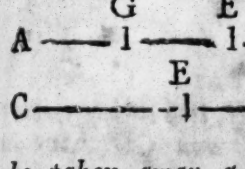
If two magnitudes AB and CD, have to one another a given reaſon, and that from each of them be taken away a given magnitude (to wit, from the magnitude AB the magnitude AE, and from the magnitude CD the magnitude CF) the remaining magnitudes EB and FD, either ſhall have to one another, a given reaſon, or the one of them ſhall be greater than the other by a given magnitude, and in reaſon.

Demonſtr. For ſeeing that each magnitude AE and CF is given, the reaſon of AE to CF is given; and if it be the ſame with that of AB to CD, that of the remainder EB to the
a 19. 5. remainder FD, *a* ſhall be alſo the ſame; and therefore the reaſon of the ſaid EB to FD ſhall be alſo given.

But if it be not the ſame, let it be as AB to CD, ſo AC to CF. Now the reaſon of AB to CD is given, there-

therefore alſo that of AG to CF ſhall be given.
But CF is given, therefore *b* AG is given. b 2. *prop.*
But AE is alſo given, therefore *c* the reſt EG c 4. *prop.*
is given; and ſeeing that as AB is to CD, ſo the part cut off AG is to the part cut off CF, and ſo alſo is *d* the reſt GB to the reſt FD; the d 4. *prop.*
reaſon of the ſaid GB to FD is alſo given. Therefore ſeeing that EG is given, EB is greater than FD *e* by a given magnitude, and e 11. *def.*
in reaſon.

PROP. XVI.



If two magnitudes AB and CD, have to one another a given reaſon and that from one of them, to wit, CD, there be taken away a given magnitude DE, and to the other AB there be added a given magnitude BF, the whole AF ſhall be greater than the reſt CE, by a given magnitude, and in reaſon.

Demonſtr. For ſeeing that the reaſon of AB to CD is given, let the ſame be made of BG to DE: Therefore *a* the reaſon of the ſaid BG a 2. *def.*
to DE is given. But DE is given, therefore *b* BG is alſo given. But BF is alſo given, b 2. *prop.*
therefore *c* the whole GF is given. And ſee- c 3. *prop.*
ing that as AB is to CD, ſo the part cut off BG, is to the part cut off DE; and *d* ſo alſo d 19. 5.
is the remainder AG to the remainder CE; the reaſon of the ſaid AG to CE is given: But GF is given, therefore the magnitude AF is greater than the magnitude CE by a given magnitude, and in reaſon.

PROP. XVII.

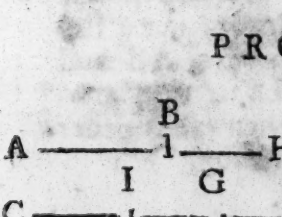
If there be three magnitudes AB, E, and CD, and that the first AB be greater than the second E, by a given magnitude, and in reason. But the third CD be also greater than the same second E, by a given magnitude, and in reason; the first AB shall have to the third CD, either a given reason, or else the one shall be greater than the other by a given magnitude, and in reason.

Demonstr. For seeing that AB is greater than E by a given magnitude, and in reason, let the magnitude AF be taken away: Therefore the reason of the remainder FB to E is given. Again, seeing that CD is greater than the said E by a given magnitude, and in reason, let the given magnitude CG be cut off therefrom; and the reason of the remainder GD to E shall be given: Therefore *a* the reason of FB to GD shall be also given. But to the said FB and GD are added the given magnitudes AF and CG: Therefore the whole AB and CD *b* shall either have to one another a given reason, or the one shall be greater than the other by a given magnitude, and in reason.

8. *prop.*

b 14. *prop.*

PROP. XVIII.



If there be three magnitudes AB, CD, and EF, and that the one of them, to wit, CD, be greater than either of the other AB or EF, by a given magnitude, and in reason; either the two others AB and EF, shall have

have to one another a given reaſon, or the one ſhall be greater than the other by a given magnitude, and in reaſon.

Demonſtr. Foraſmuch as the magnitude CD is greater than the magnitude AB by a given magnitude, and in reaſon, let the given magnitude DG be taken there-from: Therefore the reaſon of the remainder CG to AB is given. Let the ſame be made of GD to BH, therefore the reaſon of the ſaid DG to BH is given. But DG is given, therefore *a* BH is alſo given. And ſeeing that as CG is to AB, ſo is GD to BH, *b* ſo alſo is the whole CD to the whole AH, the reaſon of the ſaid CD to AH ſhall be alſo given.

a 2. *prop.*

b 12. 5.

Again, ſeeing that the ſame CD is greater than EF by a given magnitude, and in reaſon; let the magnitude DI be cut off there-from: Therefore the reaſon of the remainder CI to EF is given: Let the ſame be made of DI to FK. Therefore the reaſon of the ſaid DI to FK ſhall be alſo given. But DI is given, therefore FK is alſo given. And ſeeing that as CI is to EF, ſo is ID to FK; ſo alſo is the whole *c* CD to the whole EK; the reaſon of the ſaid CD to EK ſhall be given. But the reaſon of the ſame CD to AH is alſo given: Therefore *d* the reaſon of the ſaid AH to EK ſhall be given. And ſeeing that from the ſaid AH and EK, the given magnitudes BH and FK are cut off, the magnitudes AB and EF *e* are either in a given reaſon to one another, or the one is greater than the other by a given magnitude, and in reaſon.

c 12. 5.

d 8. *prop.*

e 15. *prop.*

PROP.

PROP. XIX.

If there be three magnitudes AB, CD, and E, and that the first AB, be greater than the second CD, by a given magnitude, and in reason; and that the second CD be greater than the third E, by a given magnitude, and in reason; also the first magnitude AB shall be greater than the third E, by a given magnitude, and in reason.

Demonstr. For seeing that CD is greater than E by a given magnitude, and in reason; let the given magnitude CF be taken there-from: Therefore the reason of the remainder FD to E is given. Again, seeing that AB is greater than the same CD by a given magnitude, and in reason: Let the magnitude AG be taken there-from: Therefore the reason of the remainder GB to CD is given: Let the same be made of GH to CF: Therefore the reason of the said GH to CF is given. But CF is given: Therefore also GH is given, and then AG is also given, the whole [a] AH shall be also given. But as GB is to CD, so is GH to CF, and so also [b] the remainder HB to the remainder FD: Therefore the reason of the said HB to FD is given. But the reason of the same FD to E is also given: Therefore the reason of HB to E is in like manner given, and so is also the magnitude AE: Wherefore the magnitude AB [c] is greater than E by a given magnitude, and in reason.

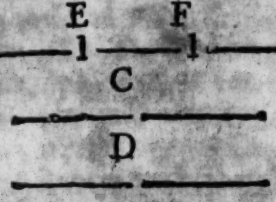
a 3. *prop.*

b 19. 5.

c 11. *def.*

OTHER

OTHERWISE.



Conſtruction. Let there be three magnitudes AB, C, and D, and let AB be greater than C by a given magnitude, and in reaſon; but let C be alſo greater than D, by a given magnitude, and in reaſon: I ſay that AB is greater than D by a given magnitude, and in reaſon.

Demonſtr. Foraſmuch as AB is greater than C by a given magnitude, and in reaſon, let the given magnitude AE be cut off there-from: Therefore the reaſon of the remainder EB to C is given. But the magnitude C is greater than the magnitude D by a given magnitude, and in reaſon; therefore *d* EB is greater than D by a given magnitude, and in reaſon: Wherefore let the given magnitude EF be cut off there-from; and the reaſon of the remainder FB to D ſhall be given. But AF is *e* given. Therefore *f* AB is greater than D by a given magnitude, and in reaſon.

d 13. *prop.*

e 3. *prop.*
f 11. *def.*

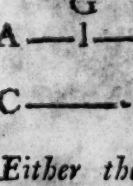
PROP. XX.

If there be two given magnitudes, AB and CD, and that from them there be taken magnitudes AE and CF, having to one another a given reaſon; either the remaining magnitudes EB and FD, ſhall have to one another given reaſons; or elſe the one ſhall be greater than the other by a given magnitude, and in reaſon.

Demonſtr.

Demonſtr. For ſeeing that both the magnitudes AB and CD, are given, the reaſon of the
a 1. *prop.* ſaid AB to CD is *a* alſo given; and if it be the ſame as of AE to CF, that of the remainder
b 19. 5. EB to the remainder FD ſhall be *b* alſo the ſame; and therefore the reaſon of the ſaid EB to FD ſhall be alſo given. But if it be not the ſame, let it be ſo as that AE be to CF, as AG to CD. Now the reaſon of the ſaid AE to CF is given: Therefore the reaſon of the ſaid AG to CD is given. But CD is
c 3. *prop.* given, therefore *c* AG is alſo given. But the
d 4. *prop.* whole AB is likewiſe given, therefore *d* the remainder BG is given. And ſeeing that as AE is to CF, ſo is AG to CD, and alſo the remainder EG to the remainder FD, the reaſon of the ſaid EG to FD is given. But GB is alſo given: Therefore the magnitude EB is
e 11. *def.* greater *e* than the magnitude BD by a given magnitude, and in reaſon.

PROP. XXI.



If there be two magnitudes given AB and CD, and to them be added other magnitudes BE and DF, having to one another a given reaſon: Either the whole AE and CF ſhall have to one another a given reaſon, or elſe the one ſhall be greater than the other by a given magnitude, and in reaſon.

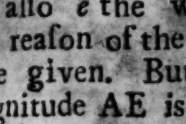
Demonſtr. For ſeeing that both the magnitudes
a 1. *prop.* AB and CD are given, their reaſon *a* is alſo given; and if it be the ſame reaſon as of BE to DF, the reaſon of the whole AE to the whole CF ſhall be alſo given; for it ſhall
b 12. 5. be *b* the ſame. But if it be not the ſame, let it be as BE is to DF, ſo BG to CD: Therefore the reaſon of the ſaid BG to CD is given.

But

But CD is given, therefore *c* alſo BG ſhall be given. But the whole AB is given, therefore alſo the *d* remainder AG ſhall be given. And ſeeing that as BE is to DF, ſo is BG to CD, and alſo *e* the whole GE to the whole CF, the reaſon of the ſaid GE to CF ſhall be likewiſe given. But AG is given, therefore the magnitude AE is greater than the magnitude CF by a given magnitude, and in reaſon.

c 2. *prop.*
d 4. *prop.*
e 12. 5.

PROP. XXII.

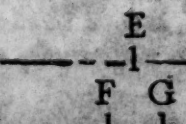


If two magnitudes AB and BC, have to ſome other magnitude D, a given reaſon, alſo their compound magnitude AC, ſhall have to the ſame magnitude D, a given reaſon.

Demonſtr. For ſeeing that each magnitude AB and BC, hath a given reaſon to D, the reaſon *a* of AB to BC is given; and by compounding, *b* the reaſon of AC to BC is given. But that of BC to D is alſo given, therefore *c* the reaſon of the ſaid AC to D ſhall be likewiſe given.

a 8. *prop.*
b 6. *prop.*
c 8. *prop.*

PROP. XXIII.



If the whole AB be to the whole CD in a given reaſon, and that the parts AE and EB, be to the parts CF and FD in given reaſons, altho' they be not the ſame, the whole (to wit, AB, AE, and BE,) ſhall be to the whole (to wit, CD, CF, and FD,) in given reaſons.

Demonſtr. For ſeeing that AE is to CF in a given reaſon, let the ſame be made of AB to CG; therefore the reaſon of the ſaid AB

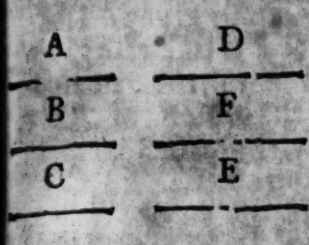
to

a 19. 5. to CG is given; and consequently, also that a of the rest EB to the rest FG. But the reason of FD to the same EB is also given: There-
b 8. *prop.* fore the reason of FD to FG *b* is likewise given;
c 5. *prop.* and therefore *c* that of FD to the remainder GD is also given. But the reason of AB to each of the magnitudes CD and CG is given:
d 8. *prop.* Therefore *d* also the reason of CD to CG is
e 5. *prop.* given, and again *e* that of CD to the remainder GD. But the reason of FD to DG is
f 8. *prop.* given, therefore also *f* that of the same CD to FD, and consequently
g 5. *prop.*

that of *g* CD to the remainder FC; and therefore also the reason of CF to FD shall be given. But the reason of EB to FD is proposed to be given; therefore the reason of CF to EB shall be given. Again, for that the reason of AB to CD is given; and also that of CD to each of those FC and FD, the reason of the same AB to
h 8. *prop.* each of the said FC and *h* FD, shall be likewise given. But the reason of the said FD to EB is given: Therefore the reason of AB to BE shall be also given, and consequently
i 5. *prop.* AB to the remainder *i* AE. Wherefore by
k *sch.* 5. *pr.* division *k* the reason of AE to EB shall be likewise given. But the reason of EB to FD is given. Therefore also that of AE to FD. In like manner, seeing that the reason of CD to AB is given; and that of AB to each of his parts AE and EB; also the reason of the said
l 8. *prop.* CD to each of the said AE and EB, *l* shall be given: Wherefore each of the magnitudes AB, CD, AE, EB, CF, and FD, is to each of the others in a given reason.

PROP.

PROP. XXIV.



If of three right lines A, B, and C, proportional, A to B, as B to C, the first A hath to the third C a given reason, it will also have to the second B a given reason.

Demonstr. For, let there be exposed another right line D, and seeing that the reason of A to C is given: Let the same be made of D to F; therefore the reason of D to F is given. But D is given, therefore F is also given; betwixt the two right lines D and F, let there be taken *a* a mean proportional E. Therefore the rectangle made under D and F is equal *b* to the square of E. But the same rectangle of D and F is *c* given: (for all the angles of that rectangle are given, being right angles, and the reasons that the sides have to one another are also given;) therefore the square of E is given, and consequently the same right line E is also given (for one equal thereto may be found, *d* seeing that the rectangle of D and F is given.) But D is given, therefore *e* the reason of D to E is given, and as A is to C, so D is to F. But as A is to C, *f* so the square of A is to the rectangle of A and C, and also as D is to F, so the square of D is to the rectangle of D and F. Therefore as the square of A is to the rectangle of A and C, so the square of D is to the rectangle of D and F. But the rectangle of A and C is equal to the square of B, (seeing that A, B, and C, are proportional) and that of D and F to the square of E, therefore as the square of A is to the square of B, so the square of D is to the square of E:

a 13. 6.
b 17. 6.
c 3. *def.*
d 14. 2.
e 1. *prop.*
f 1. 6.

Where-

g 12. 6. Wherefore *g* as A is to B, ſo D is to E. But
h 2. *def.* the reaſon of D to E is given, therefore
alſo the reaſon of A to B is given.

OTHERWISE.

Demonſtr. Foraſmuch as the reaſon of A to C is given, and that as A is to C, ſo the ſquare of A and C, the reaſon of the ſaid ſquare of A to the rectangle made of A and C, is alſo given. But to that rectangle made of A and C the ſquare of B is equal (ſeeing that A, B, and C, are proportional;) therefore the reaſon of the ſquare of A to the ſquare of B is given; and by conſequence, the reaſon of the line A to the line B is given; for to each of them A and B, we have exhibited an equal to the proper ſquare of each one.

A
B
C

PROP. XXV.

If two lines AB and CD, given by poſition do interſect the point E in which they interſect one another, is given by poſition.

Demonſtr. For if it change its place, the one or the other of the lines AB and CD, would change its poſition: But ſo it is that by Supa 4. *def.* poſition it changeth not: Therefore *a* the point E is given by poſition.

PROP. XXVI.

A ——— B *If the extremities A and B, of a right line AB, be given by poſition, that ſame right line AB is given by poſition and by magnitude.*

Demonſtr.

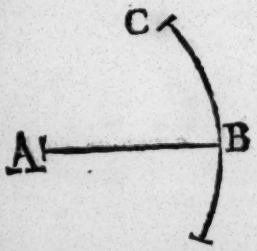
Demonſtr. For if the point A remaining in its place, the poſition, or the magnitude of the right line AB ſhall change, the point B will fall elſewhere. But ſo it is, that by Suppoſition it doth not fall elſewhere. Therefore the right line AB is given by poſition, and by magnitude.

PROP. XXVII.

A——B *If one of the extremities A of a right line AB, given by poſition and magnitude be given, the other extremity B ſhall be alſo given.*

Demonſtr. For if the point A remaining in its place, the point B ſhall change and fall in ſome other place, either the poſition of the right line AB, or its magnitude would change: But ſo it is that according to the Suppoſition, neither the one nor the other doth change. Therefore the point B is given.

OTHERWISE.



Conſtr. On the center A, with the diſtance AB, deſcribe the circumference BC.

Demonſtr. Therefore *a* that circumference BC is given by poſition. But the right line AB is alſo given by poſition; therefore the point *b* B is given. *a* *a* 6. *def.* *b* 25. *prop.*

Wherefore *g* as A is to B, ſo D is to E. But
g 12. 6. *h* 2. *def.* the reaſon of D to E is given, therefore
alſo the reaſon of A to B is given.

OTHERWISE.

Demonſtr. For aſmuch as the reaſon of A
C is given, and that as A is to C, ſo
ſquare of A and C, the reaſon of the ſaid ſquare
of A to the rectangle made of A and C, is alſo
given. But to that rectangle made of A and C
the ſquare of B is equal (ſeeing that
A, B, and C, are proportional;)
therefore the reaſon of the ſquare
of A to the ſquare of B is given,
and by conſequence, the reaſon of
that line A to the line B is given;
for to each of them A and B, we
have exhibited an equal to the
proper ſquare of each one.

A
B
C

PROP. XXV.

If two lines AB and CD, given by poſition do manifeſt the point E in which they interſect one another, is given by poſition.

Demonſtr. For if it change its place, the one
a 4. *def.* the other of the lines AB and CD, would
change its poſition: But ſo it is that by Sup-
poſition it changeth not: Therefore *a* the point
E is given by poſition.

PROP. XXVI.

A ——— B *If the extremities A and B, of a right line AB, be given by poſition, that ſame right line AB is given by poſition and by magnitude.*

Demonſtr.

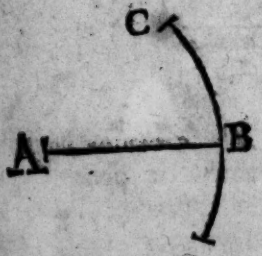
Demonſtr. For if the point A remaining in its place, the poſition, or the magnitude of the right line AB ſhall change, the point B will fall elſewhere. But ſo it is, that by Suppoſition it doth not fall elſewhere. Therefore the right line AB is given by poſition, and by magnitude.

PROP. XXVII.

A——B *If one of the extremities A of a right line AB, given by poſition and magnitude be given, the other extremity B ſhall be alſo given.*

Demonſtr. For if the point A remaining in its place, the point B ſhall change and fall in ſome other place, either the poſition of the right line AB, or its magnitude would change: But ſo it is that according to the Suppoſition, neither the one nor the other doth change. Therefore the point B is given.

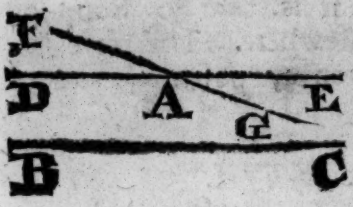
OTHERWISE.



Conſtr. On the center A, with the diſtance AB, deſcribe the circumference BC.

Demonſtr. Therefore *a* that circumference BC is given by poſition. But the right line AB is alſo given by poſition; therefore the point *b* B is given. *a* 6. *def.* *b* 25. *prop.*

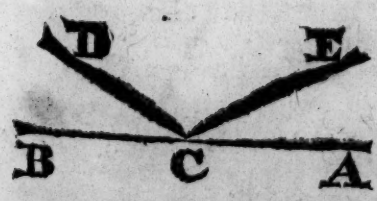
PROP. XXVIII.



If by a given point A, there be drawn a right line DAE, against another right line BC, given by position, the right line DAE ſo drawn, is given by poſition.

Demonſtr. For if it be not given, the point A remaining in its place, the poſition of the right line DAE may change: Let it then change if it be poſſible, and fall elſewhere, remaining parallel to BC, and let it be the line FAG: Therefore BC is parallel to the ſaid line FAG. But
a 13. *def.* *a* the ſame BC is alſo parallel to DAE:
b 30. 1. Therefore *b* DAE is parallel to the ſaid line FAG, which is abſurd; ſeeing they joyn together, and meet in A: Therefore the poſition of the right line DAE falls not elſewhere. Wherefore the ſaid line DAE is given by poſition.

PROP. XXIX.

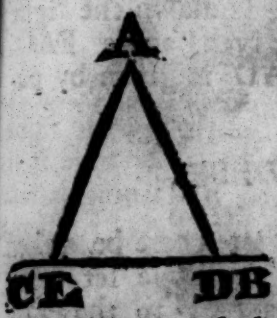


If to a right line AB, given by poſition, and to a point C given therein, there be drawn a right line CD, which ſhall make a given angle ACD, the line drawn CD, is given by poſition.

Demonſtr. For if it be not given by poſition, the point C remaining in its place, the poſition of the line CD obſerving the magnitude of the angle ACD, will fall elſewhere. Let it fall elſewhere then if it be poſſible, and let it

it be CE. Therefore the angle ACD is equal to the angle ACE, the greater to the leſſer, which is abſurd. Therefore the poſition of the right line CD, ſhall not fall elſewhere; and therefore the ſaid line CD is given by poſition.

PROP. XXX.



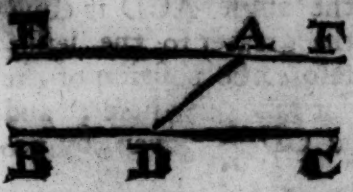
If from a given point A, be drawn to a right line BC, given by poſition, a right line AD, making a given angle ADB, that line drawn AD is given by poſition.

Demonſtr. For if it be not given, the point A remaining in its place, the poſition of the right line AD keeping, magnitude of the angle ADB, will change. Let it change then, and let it be the right line AE: Therefore the angle ADB is equal to the angle AEB, the greater *a* to the leſſer, which is abſurd. Therefore the poſition of the right line AD doth not change; and therefore the ſaid line AD is given by poſition. a 16. 1.

OTHERWISE.

Conſtr. By the point A let there be drawn the line EAF, parallel to the right line BC.

Demonſtr. Then ſeeing that by the given point A, and againſt the right line BC, given by poſition, there is drawn the right line EF, thoſe lines EF and BC are parallels. But on



the ſame lines doth alſo fall the right line AD. Therefore *b* the angle FAD is equal to the given angle ADB; and therefore it is alſo given. Wherefore to the right line EF given by poſition, and to the given point A therein, there is drawn the right line AD, making the given angle FAD. Therefore *c* the ſaid line AD is given by poſition.

b 29. 1.

c 29. *prop.*

OTHERWISE.

Conſtr. In the line BCE, let there be taken the given point C, and by the ſame let there be drawn the line CF, parallel to the ſaid DA.

Demonſtr. Foraſmuch as AD and FC are parallels, and that on them there doth fall the right line BCE, the angle FCB is equal *d* to the given angle ADB; and therefore it is alſo given. And ſeeing that the right line BC is given by poſition, and that to a given point C therein, there is drawn the right line FC, making the given angle FCB, that ſame line FC *e* is given by poſition. But by the given point A, oppoſite to the line FC given by poſition, there is drawn the line AD. Therefore the ſaid line *f* AD is given by poſition.

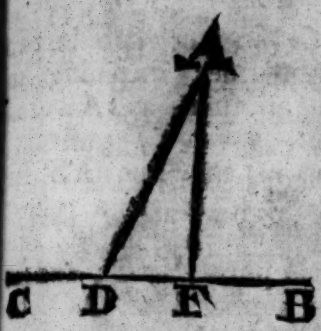
d 29. 1.

e 29. *prop.*

f 28. *prop.*

OTHER-

OTHERWISE.



Conſtr. In the right line BC aſſume ſome point at F, and draw AF.

Demonſtr. Foraſmuch as each point A and F is given, the right line AF is given *g* by poſition. But the line BC is alſo given by poſition. Therefore * the angle AFD is given. But by ſuppoſition, the angle ADF is given: Therefore DAF (which is the reſidue *h* of two right angles) is given; and ſeeing that to the right line AF given by poſition, and to the given point therein A there is drawn the right line DA, making the given angle DAF, *i* that ſame line DA is given by poſition.

g 26. *prop.*

h 32. 1.

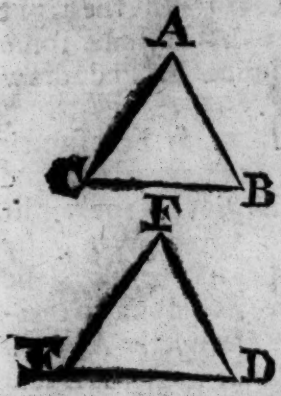
i 25. *prop.*

Scholium.

* *EUCLIDE ſuppoſeth here that two right lines being given by poſition, and inclining to one another do make a given angle, which ſome do demonſtrate after this manner.*

Demonſtr. Foraſmuch as the two right lines given by poſition, do incline to one another, the inclination of thoſe lines is given. But the angle is the inclination of the lines: Therefore the angle which makes the right lines given by poſition, and inclining to one another, is given.

Another thus demonſtrateth it.



Conſtr. Let there be two right lines inclining to one another, as AB and CB, given by poſition, and in the line AB let there be taken a given point A, and in BC alſo ſome point, as C; and let the right line AC be drawn.

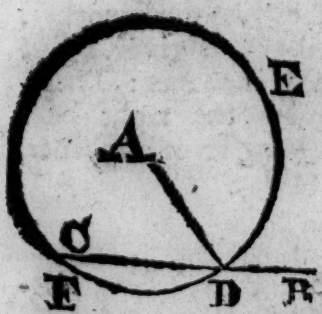
Demonſtr. Seeing that as well the point B, as each of the points A and C is given, *k* the three right lines AB, BC, and AC, are given by magnitude. Wherefore of three direct lines equal unto them, a triangle may be conſtituted: Let there then be made the triangle FDE, having the ſide FD equal to the ſide AB, the ſide FE equal to the ſide AC, and the baſe DE equal to the baſe BC.

k 26. *prop.*

Seeing then the angles compriſed of equal right lines are equal, we have found the angle FDE equal to the angle ABC; and therefore the ſame *l* angle ABC is given.

l 1. *def.*

PROP. XXXI.



If from a given point A there be drawn to a right line given by poſition BC, a right line AD, given by magnitude, that line AD ſhall be alſo given by poſition.

Conſtr. From the center A, with the diſtance AD, let the circle DEF be deſcribed.

Demonſtr.

Demonſtr. Foraſmuch as the center A is given by poſition, and the ſemidiameter AD by magnitude, the circle DEF *a* is given by poſition. But the right line BC is alſo given by poſition: Therefore the point of interſection D *b* is given, and ſeeing that the point A is alſo given: *c* the right line AD is given by poſition.

a 6. *def.*
b 25. *prop*
c 26. *prop.*

PROP. XXXII.

If unto parallel right lines AB and CD, given by poſition, there be drawn a right line EF, making the given angles BEF and EFD, the line drawn EF ſhall be given by magnitude.

Conſtr. For let there be taken in the line CD a given point G, and from that point let be drawn GH parallel to FE.

Demonſtr. Foraſmuch as the lines EF and HG are parallels, and that on them doth fall the line CD; *a* the angle EFD is equal to the angle FGH. But the angle EFD is given, therefore the angle FGH is alſo given. And foraſmuch as to the right line CD given by poſition, and to the point G given in the ſame, there is drawn the right line GH, making the given angle FGH, *b* the ſaid line GH is given by poſition. But AB is alſo given by poſition, therefore *c* the point H is given. But the point G is alſo given: Therefore *d* the line GH is given by magnitude, and is *e* equal to EF. Wherefore *f* the ſaid line EF is given by magnitude.

a 29. 1.
b 29. *prop*
c 25. *prop.*
d 26. *prop.*
e 34. 1.
f 1. *def.*

PROP. XXXIII.

If unto parallel right lines AB and CD, given by position, there be drawn a right line EF, given by magnitude, that line EF ſhall make the given angles BEF and DFE.

Conſtr. For let there be taken in the right line AB the point G, and by that point let there be drawn the line GH parallel to EF.

Demonſtr. Therefore EF
a 34. 1. is equal to the ſaid *a* GH. But EF is given by magnitude, therefore GH is alſo given by magnitude. But the point G is given, and therefore if on that point, with the diſtance
b 6. *def.* GH, there be deſcribed a circle, *b* that circle ſhall be given by poſition: Let it be then deſcribed, and let it be HKL, the ſaid circle HKL is therefore given by poſition. But the line CD which doth cut the circumference KHL in H, is alſo given by poſition. Therefore
c 25. *prop.* the ſaid point of interſection H *c* is given.
d 26. *prop.* But the point G is given: Therefore *d* the right line GH is given by poſition. But the right line CD is alſo given by poſition: There-
e *ſch.* 30. *prop.* fore *e* the angle GHF is given. But to that angle *f* the angle EFD is equal: Therefore
f 29. 1. the angle EFD is given; and therefore alſo the angle BEF; for that it is the reſidue of
g 29. 1. the ſumm of two *g* right angles.

OTHERWISE.

Conſtr. Let there be taken in the right line CD the point G, and let GD be put equal to EF,

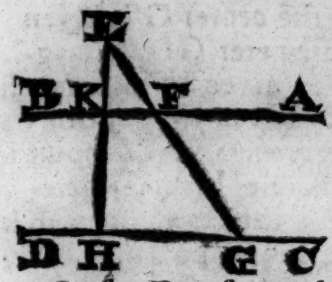
EF, then from the center G, with the diſtance GD, let there be deſcribed the circle DBH, and draw GB.

Demonſtr. Foraſmuch as the center G is given by poſition, and the ſemidiameter GD by magnitude, the circle BDH *h* is given by poſition. But the line AB is alſo given by poſition: Therefore *i* the point B is given. But the point G is alſo given, therefore *k* the right line GB is given by poſition. But the right line CD is alſo given by poſition: Therefore *l* the angle BGD is given. Wherefore if EF be parallel to BG, the angle EFD *m* ſhall be given, and conſequently alſo the other angle BEF. But the right lines BG and EF being not parallels, let them meet in the point H. Foraſmuch as EB is parallel to FG, and EF is equal to GD, that is to ſay, to BG; alſo FH *n* ſhall be equal to GH (for EH and BH being cut proportionally *o* by the parallel FG, as EF is to FH, ſo is BG to GH; and by permutation, as EF is to BG, ſo is FH to GH:) Therefore *p* the angle HFG is equal to the angle HGF, but the ſaid angle HGF is given (for that it is equal *q* to the given angle BGD:) Therefore the angle HFG is alſo given. But to that angle the angle BEF is equal; and therefore is given, as alſo the remaining angle EFG.

h 6. *def.*
i 25. *prop.*
k 26. *prop.*
l *ſch.* 30. *prop.*
m 29. 1.
n 14. 5.
o 2. 6.
p 5. 1.
q 15. 1.

PROP.

PROP. XXXIV.



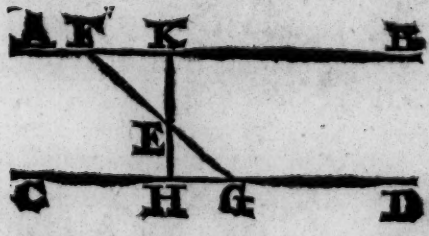
If from a given point E, there be drawn unto parallel right lines AB and CD, given by position, a right line EFG, that right line EFG shall be divided in a given reason (to wit) as EF to FG.

Constr For from the point E let there be drawn the line EH, perpendicular to the line CD.

Demonstr. Forasmuch as from the given point E there is drawn to the line CD the right line EH, making the given angle EHG *a* the said line EH is given by position, but both the one and the other lines AB and CD is also given by position. Therefore *b* the points of intersection K and H, are given. But the point E is also given: Therefore *c* each line EK and KH is given. Wherefore *d* the reason of the said EK to KH is given. But as EK is to KH, so is EF to FG; (for in the triangle GEH the line KF being parallel to HG, the sides EH and EG are cut proportionally:) Therefore the reason of the said EF to FG is given.

a 30. *prop.*
b 25. *prop.*
c 26. *prop.*
d 1. *prop.*

OTHERWISE.



Constr. To the parallel right lines given by position, AB and CD, let there be drawn from the point E the right line FEG: I say that the reason of GE to EF is given.

Demonstr.

Demonſtr. For from the point E let there be drawn to CD the perpendicular EH, and produced to the point K; ſeeing therefore that from the point E to the right line CD, given by poſition, there is drawn the line EH, making the given angle EHG, *a* the ſaid line EH is given by poſition. But each line AB and CD is alſo given by poſition: Therefore *b* each point of interſection H and K is given. But the point E is alſo given, therefore *c* each of the lines EH and EK is given by magnitude; and therefore *d* the reaſon of the ſaid EH to EK is given. But *e* as EH is to EK, ſo is EG to EF (for the oppoſite angles at the point E being equal, and the lines AB and CD parallels, the triangles EHG and EKF are equiangled; and therefore as EH is to EG, ſo is EK to EF; and by permutation as EH to EK, ſo is EG to EF.) Therefore the reaſon of the ſaid lines EG to EF is given.

a 30. *prop.*
b 25. *prop.*
c 26. *prop.*
d 1. *prop.*
e 4. 6.

PROP. XXXV.

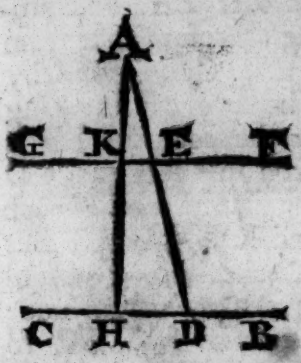
If from a given point A, to a right line BC, given by poſition, there be drawn a right line AD, which let be divided in E, in a given reaſon (to wit) as AE to ED, and that by the point of ſection E, there be drawn a right line FEG, oppoſite to the right BC, given by poſition, the line FG drawn ſhall be given by poſition.

Conſtr. For from the point A, let there be drawn the line AH, perpendicular to the line BC.

Demonſtr. For ſeeing that from the given point A there is drawn to BC given by poſition, the right line AH making the given angle AHD, *a* the ſaid line AH is given by poſition. But BC is alſo given by poſition: Therefore *b* the point H is given. But the point A is alſo given:

a 30. *prop.*
b 25. *prop.*

e 26. *prop.* given: Therefore *c* the line AH is given by magnitude and by position. And seeing that



d 2. 6. *d* as AE is to ED, so is AK to KH, and that the reason of AE to ED is given, also the reason of AK to KH is given; and by compounding, *e* the reason of AH to AK is given. But AH is given by magnitude: Therefore *f* also AK is given by magnitude. But AK is also given by position, and the point A is given: Therefore *g* the point K is also given, and seeing that by the said given point K there is drawn the line FG, opposite to the right line BC given by position; the said line FG *h* is given by position.

e 6. *prop.*

f 2. *prop.*

g 27. *prop.*

h 28. *prop.*

PROP. XXXVI.

If from a given point A, there be drawn to a right line BC given by position, a right line AD, and to it be added a right line AE, having to the same AD a given reason, and that by the extremity E of the added line AE, there be drawn a right line FEK, opposite to the line BC, given by position, that same line FEK shall be given by position.

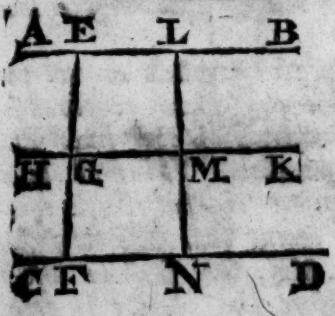
Constr. For from the point A let there be drawn to the line BC, the perpendicular AL, and let it be prolonged to the point G.

Demonstr. Forasmuch as from the given point A, there is drawn to the right line BC, given

given by position, the right GL, which makes the given angle GLD, *a* that line GL is given by position. But BC is also given by position, therefore *b* the point L is given; and seeing that the point A is also given, the line *c* AL is given. But forasmuch as the reason of AE to AD is given, and that *d* as the said AE is to AD, so is AG to AL; (because the triangles ALD and AGE are equiangled) the reason of AG to AL is also given. But AL is given by magnitude: Therefore *e* AG is given by magnitude. But it is also given by position, and the point A is given: Therefore *f* the point G is also given. And seeing that by the same given point G, there is drawn the line FK, opposite to the right line BC, given by position, *g* the said line FK is given by position.

a 30. *prop.*
b 26. *prop.*
c 26. *prop.*
d 4. 6.
e 2. *prop.*
f 27. *prop.*
g 28. *prop.*

PROP. XXXVII.



If unto parallel right lines AB and CD, given by position, there be drawn a right line EF, divided in the point G, in a given reason, (to wit, of EG to GF;) but if by the point of section G, there be drawn opposite to the right lines AB or CD, given by position, a right line HGK, that line drawn shall be given by position.

Constr. For let there be taken in the line AB the given point L, and from that point let there be drawn the line LN, perpendicular to CD.

Demonstr. Seeing that from the given point L, there is drawn to the right line CD, the line LN, making the given angle LND, the said

a 30. *prop.* said LN *a* is given by position. But CD is also given by position: Therefore the point
b 25. *prop.* N *b* is given. But the point L is also given:
c 26. *prop.* Therefore *c* the line LN is given; and seeing that the reason of FG to GE is given, and that * as FG is to GE, so is NM to ML, the reason of the said MN to ML is given;
d 6. *prop.* and in compounding, *d* the reason of LN to LM is also given. But LN is given by mag-
e 2. *prop.* nitude, therefore ML is *e* given by magnitude. But it is also given by position, and the point
f 27. *prop.* L is given: Therefore the point M *f* is also given. And considering that by the said point M there is drawn the right line KH, opposite to the right line CD, given by position, the said line KH is also given by position.

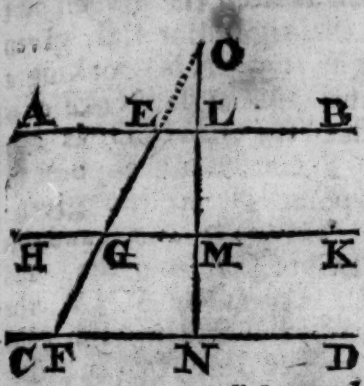
Scholium.

* *EUCLIDE supposeth here, that as FG is to GE, so NM is to ML; but by another it is thus demonstrated.*

The lines EF and LN are parallels or not parallels: Let them in the first place be parallels, and forasmuch as by Construction the lines EL, FN, EF, and LN, are parallels, EN shall be a parallelogram; and therefore the side EF is equal to the side LN. Again, seeing that MG is parallel to FN, and GF to MN, GN shall be also a parallelogram; and therefore the side GF is equal to the side MN. Wherefore the equal sides EF and LN, shall have to the equal sides FG and MN, g *one and the same reason. Therefore as*
g 7. 5. *EF is to FG, so is LN to MN; and in dividing,*
h 17. 5. h *as GE to GF, so is LM to MN.*

Now

Now ſuppoſe that the lines EF and LN be not parallels, but that they meet in the point O. For-aſmuch as in the triangle OFN there is drawn HK, parallel to FN one of the ſides; i the ſides OF and ON are divided proportionably; and therefore as FG is to GO, ſo is NM to MO. Again, ſeeing that in the triangle OGM there is drawn EL, parallel to the ſide GM, the ſides OG and OM are divided proportionally: Wherefore k as OE is to EG, ſo is OL to LM; and by compounding, l as OG is to EG, ſo is OM to LM; but it hath been demonſtrated that as FG is to GO, ſo is NM to MO; therefore in reaſon of equality, m as FG is to GE, ſo is NM to ML.



i 2. 6.

k 2. 6.

l 18. 5.

m 22. 5.

PROP. XXXVIII.

If unto parallel right lines AB and CD, there be drawn a right line EF, and that to it there be added ſome other right line EG, which hath a given reaſon to the ſame EF, but if by the extremity G of the added line EG, there be drawn a right line HK, againſt the parallels given by poſition AB and CD, the line drawn HK ſhall be alſo given by poſition.

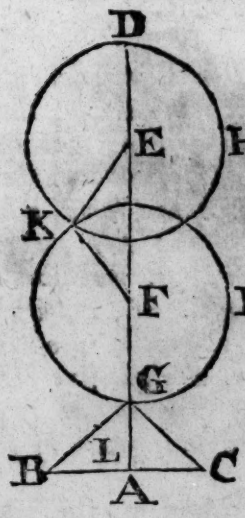
Conſtr. For let there be taken in the line AB, the given point N, and from thence let there

there be drawn to CD the perpendicular NM, and let it be prolonged to the point L.

Demonſtr. Foraſmuch as from the given point N there is drawn to the right line CD, given by poſition, the right line NM making a given angle NMF, the ſaid angle NMF *a* is given by poſition. But the line CD is alſo given by poſition: Therefore *b* the point M is given. But the point N is alſo given: Therefore *c* the line NM is given, and for that the reaſon of EG to EF is given, and that *d* as EG is to EF, ſo is LN to NM, the reaſon of LN to NM is alſo given: But NM is given, therefore LN is *e* alſo given. But the point N is given: Therefore *f* the point L is alſo given. Seeing then that by the given point L there is drawn the right line HK, oppoſite to the line AB given by poſition, *g* the ſaid line AK is alſo given by poſition.

a 30. *prop.*
b 25. *prop.*
c 26. *prop.*
d *ſch.* 37. *prop.*
e 2. *prop.*
f 27. *prop.*
g 28. *prop.*

PROP. XXXIX.



If all the ſides of a triangle ABC are given by magnitude, the triangle is given by Kind.

Conſtr. For, let there be expoſed the right line DG given by poſition, ending in the point D; but being infinite towards the other part G, and therein let there be taken DE, equal to AB.

Demonſtr. Now ſeeing the ſaid AB is given by magnitude, DE is ſo alſo; but the ſame DE is alſo given by poſition, and the point D is given: Therefore *a* the point E is given.

a 27. *prop.*

Again,

Again, Let EF be put equal to BC; and seeing that BC is given by magnitude, EF shall be so also. But the said EF is in like manner given by position, and the point E is given: Therefore *b* the point F is given. b 27. *prop.*

Furthermore, Let FG be taken equal to AC. Now forasmuch as the said AC is given by magnitude, FG is so also. But FG is also given by position, and the point F is given: Therefore the point G is also given. Now from the center E, with the distance ED, let there be described the circle DHK, *c* and that circle shall be given by position. Again, on the center F, and distance FG, let there be described the circle GLK. Therefore *d* the said circle GLK is given by position; and therefore *e* the point of intersection I is given. But each of the points E and F is given: Therefore each line EK, EF, and FK, is given by position and magnitude. Therefore the triangle EKF is given * by kind; but it is equal and alike to the triangle ABC; and therefore the triangle ABC is also given by kind. c 6. *def.* d 6. *def.* e 25. *prop.* f 26. *prop.*

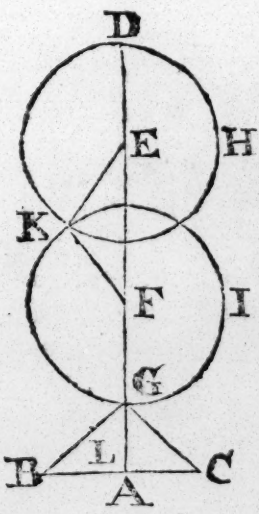
Scholium.

EUCLID supposeth here that a triangle whose sides are given by magnitude and position, is given by kind; but the antient Interpreters demonstrate it in a manner thus. Forasmuch as the right lines KE and EF are given, g 1. *prop.* *the reason which they have to one another is given. Also the right lines EF and FK being given, their reason is also given; and in like manner, the reason*

there be drawn to CD the perpendicular NM, and let it be prolonged to the point L.

Demonſtr. Foraſmuch as from the given point N there is drawn to the right line CD, given by poſition, the right line NM making a given angle NMF, the ſaid angle NMF *a* is
a 30. *prop.* given by poſition. But the line CD is alſo
b 25. *prop.* given by poſition: Therefore *b* the point M is given. But the point N is alſo given:
c 26. *prop.* Therefore *c* the line NM is given, and for that the reaſon of EG to EF is given, and
d *ſch.* 37. *prop.* that *d* as EG is to EF, ſo is LN to NM, the reaſon of LN to NM is alſo given: But NM
e 2. *prop.* is given, therefore LN is *e* alſo given. But
f 27. *prop.* the point N is given: Therefore *f* the point L is alſo given. Seeing then that by the given point L there is drawn the right line HK,
g 28. *prop.* oppoſite to the line AB given by poſition, *g* the ſaid line AK is alſo given by poſition.

PROP. XXXIX.



If all the ſides of a triangle ABC are given by magnitude, the triangle is given by Kind.

Conſtr. For, let there be expoſed the right line DG given by poſition, ending in the point D; but being infinite towards the other part G, and therein let there be taken DE, equal to AB.

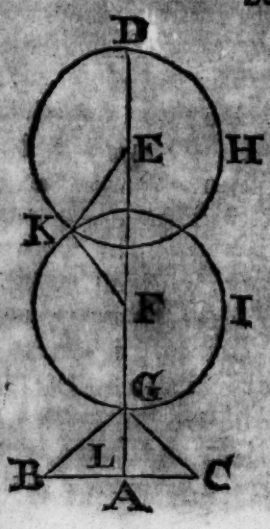
Demonſtr. Now ſeeing the ſaid AB is given by magnitude, DE is ſo alſo; but the ſame DE is alſo given by poſition, and the point
a 27. *prop.* D is given: Therefore *a* the point E is given.

Again,

Again, Let EF be put equal to BC; and ſeeing that BC is given by magnitude, EF ſhall be ſo alſo. But the ſaid EF is in like manner given by poſition, and the point E is given: Therefore *b* the point F is given. b 27. *prop.*

Furthermore, Let FG be taken equal to AC. Now foraſmuch as the ſaid AC is given by magnitude, FG is ſo alſo. But FG is alſo given by poſition, and the point F is given: Therefore the point G is alſo given. Now from the center E, with the diſtance ED, let there be deſcribed the circle DHK, *c* and that circle ſhall be given by poſition. Again, on the center F, and diſtance FG, let there be deſcribed the circle GLK. Therefore *d* the ſaid circle GLK is given by poſition; and therefore *e* the point of interſection K is given. But each of the points E and F is given: Therefore each line *f* EK, EF, and FK, is given by poſition and magnitude. Therefore the triangle EKF is given * by kind; but it is equal and alike to the triangle ABC; and therefore the triangle ABC is alſo given by kind. c 6. *def.* d 6. *def.* e 25. *prop.* f 26. *prop.*

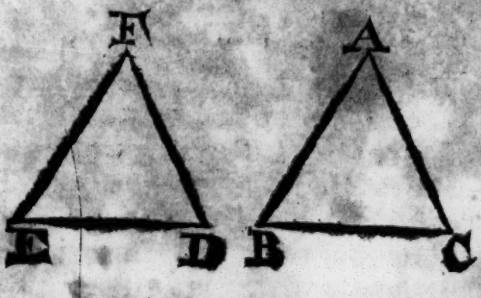
Scholium.



* *EUCLIDE ſuppoſeth here that a triangle whoſe ſides are given by magnitude and poſition, is given by kind; but the antient Interpreters demonſtrate it in a manner thus. Foraſmuch as the right lines KE and EF are given, the reaſon which they have to one another is given. Alſo the right lines EF and FK being given, their reaſon is alſo given; and in like manner, the reaſon* g 1. *prop.*

of the said EK and FK is given. Again, seeing that the same lines KE and EF are given by
h *sch.* 30. *prop.* *position,* h *the angle KEF is given by magnitude: Moreover, the right lines EF and FK being given by position, the angle EFK is given by magnitude, as is also the residue EKF, and so in the triangle EKF are all the angles given, and also the reasons of the sides:*
i 5. *def.* *Therefore* i *the said triangle EKF is given by kind.*

PROP. XL.



If the angles of a triangle ABC, are given by magnitude, the triangle is given by kind.

Constr. Let there be exposed the right line DE, given by position and by magnitude; and let there be constituted at the point D the angle EDF, equal to the angle CBA; but in the point E the angle DEF, equal to the angle BCA; therefore the third angle BAC is equal to the third angle DFE.

Demonstr. For each of the angles constituted in the points A, B, and C, is given: Therefore each of those which are posited in the points D, F, and E, is also given; and seeing that to the right line DE given by position, and to the point D given therein, there is drawn the right line DF, which makes the gi-
a 29. *prop.* ven angle EDF, *a* the line DF is given by position; and by the same reason, the line
b 25. *prop.* EF is given by position: Therefore *b* the point F is given by position. But each of the
c 26. *prop.* points D and E is given: Therefore *c* each of the

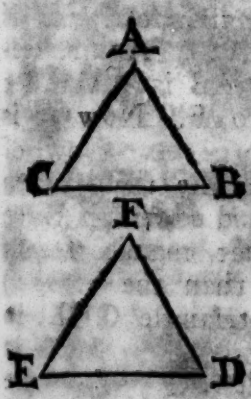
the lines DF, DE, and EF, is given by magnitude. Wherefore the triangle DFE is given by kind; and is alike to the triangle ABC: Therefore the triangle ABC is given by kind.

PROP. XLI.

If a triangle ABC, hath one angle BAC given, and that the two ſides BA and AC, which do conſtitute it, have to one another a given reaſon, the triangle is given by kind.

Conſtr. For, let there be expoſed the right line DF given by magnitude and poſition. But thereon, and at the given point F, let there be conſtituted the angle DFE equal to the angle BAC.

Demonſtr. Now the angle BAC is given: Therefore alſo the angle DFE is given, and ſeeing that to the right line DF given by poſition, and from the given point F therein, is drawn a right line FE, making the given angle DFE, *a* the ſaid line FE is given by poſition. But ſeeing that the reaſon of AB to AC is given, let the ſame be made of DF to FE, then let DE be drawn. Therefore the reaſon of DF to FE is given. But DF is given: Therefore *b* FE is given by magnitude. But the ſame FE is alſo given by poſition, and the point F is given. Therefore *c* the point E is alſo given. But each of the points D and F is given: Therefore *d* each of the right lines DF, FE, and DE, is given by poſition and magnitude. Wherefore *e* the triangle DEF is



a 29. *prop.*

b 2. *prop.*

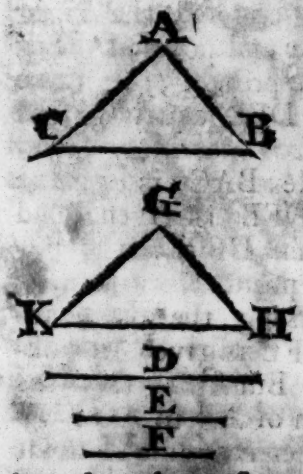
c 27. *prop.*

d 26. *prop.*

e 39. *prop.*

given by kind. And ſeeing that the two triangles ABC and DEF have an angle equal to an angle, that is to ſay, the angle BAC to the angle DFE, and the ſides which conſtitute
f 6. 6. thoſe equal angles, proportional; *f* the triangle ABC is alike to the triangle DEF. But the triangle DEF is given by kind: Therefore the triangle ABC is given by kind.

PROP. XLII.



If the ſides of a triangle ABC, be to one another in given reaſons, the triangle ABC is given by kind.

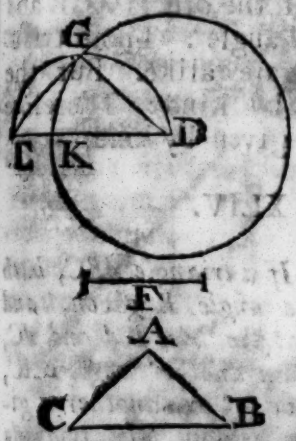
Conſtr. For, Let there be expoſed the right line D, given by magnitude, and ſeeing that the reaſon of BC to AC is given, let the ſame be made of D to E.

Demonſtr. Now D is
a 2. *prop.* given, therefore *a* E is alſo given. Again, ſeeing that the reaſon of AC to AB is given, let the ſame be made of E to F. Now E is
b 2. *prop.* given, therefore *b* F is alſo given. Now of three right lines, equal to the three given right lines D, E, and F, (and of which three lines, two of them, in what manner ſoever they be taken, are greater than the other:) Let there be conſtituted the triangle GHK, in ſuch ſort as D may be equal to HK; but E is equal to KG, and GH equal to F; therefore each of the ſaid lines HK, KG, and GH, is
c 39. *prop.* given by magnitude: Wherefore *c* the triangle HGK is given by kind. And ſeeing that as BC is to CA, ſo is D to E, and that D is equal to HK, and E to KG, as BC is to CA,

ſo

ſo HK is to KG. Again, ſeeing that as CA is to AB, ſo is E to F, and that E is equal to KG, and F to GH; as CA is to AB, ſo is KG to GH. But it hath been demonſtrated that as BC is to CA, ſo is HK to KG: Therefore by reaſon of equality, as BC is to AB, ſo is HK to GH. Therefore *d* the triangle ABC is alſo given by kind. d 5. 6.

PROP. XLIII.



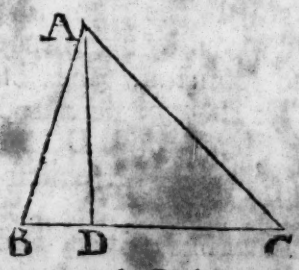
If the ſides BC and BA, about one of the acute angles of a rectangled triangle ABC, have to one another a given reaſon, that triangle is given by kind.

Conſtr. Let there be expoſed the right line DE given by magnitude and poſition, and on it let there be deſcribed the ſemicircle DGE: Therefore *a* the ſemicircle DGE is given by poſition. a 6. *def.*

Demonſtr. For the line DE being given, and divided in two equal parts, the center of the ſaid circle is given by poſition, and the ſemidiameter by magnitude. And foraſmuch as the reaſon of BC to BA is given, let the ſame be made of DE to F: Therefore the reaſon of DE to F is given. But DE is given, therefore F *b* is alſo given. Now BC is greater than *c* AB: Therefore ED is *d* alſo greater than F. Let DG be fitted equal to F, and let EG be drawn; then on the center D, with the diſtance DG, let the circle GK be deſcribed. Now that circle *e* is given by poſition, ſeeing that the center D is given, and the ſemidia- b 2. *prop.* c 19. 1. d 14. 5. e 6. *def.*

meter DG also given by magnitude. But the semicircle DGE is also given by position:
f 25. *prop.* Therefore *f* the point of intersection G is given. But the points D and E are also given, there-
g 26. *prop.* fore *g* each of the right lines DE, DG, and EG, is given by position and magnitude.
h 39. *prop.* Wherefore *h* the triangle DGE is given by kind. And seeing that the triangles ABC and DGE have an angle equal to an angle, to wit,
i 31. 3. the right angle BAC to the right angle *i* DGE, and the sides about the angles CBA and EDG proportional. But each of the others ACB and DEG are less than a right angle: Those trian-
k 7. 6. gles ABC and DEG *k* are alike. But the triangle DGE is given by kind: Therefore the triangle ABC is also given by kind.

PROP. XLIV.



If a triangle ABC, hath one angle B given, and that the sides BA and AC, about another angle BAC, have to one another a given reason, the triangle ABC is given by kind.

Constr. Now the given angle B is either acute or obtuse, (for it was a right angle in the foregoing Proposition.) Let it be in the first place acute, and from the point A let AD be drawn perpendicular to BC.

Demonstr. Therefore the angle ADB is given: But the angle B is also given; and therefore the third angle BAD is given: Wherefore
a 40. *prop.* *a* the triangle ABD is given by kind; and
b 3. *def.* therefore *b* the reason of BA to AD is given. But the reason of the same BA to AC is also
c 8. *prop.* given: Therefore *c* the reason of AD to AC is given, and the angle ADC is a right angle:

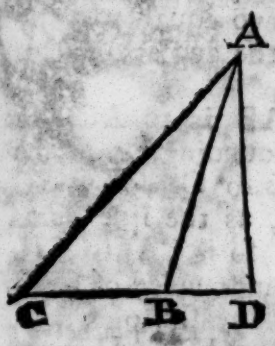
Where-

Wherefore the triangle *d* ACD is given by kind: Therefore *e* the angle C is given. But the angle B is alſo given; and therefore the other angle BAC is given: Therefore *f* the triangle ABC is given by kind.

d 43. *prop.*
e 3. *def.*
f 40. *prop.*

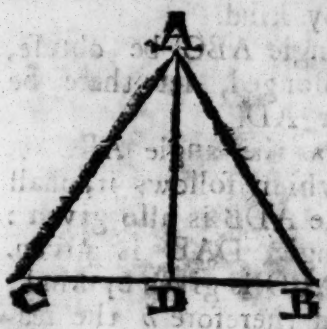
Conſtr. Now let the angle ABC be obtuſe, and on the ſide CB prolonged, let there be drawn the perpendicular AD.

Demonſtr. Foraſmuch as the angle ABC is given, the angle ABD which follows it, ſhall be given. But the angle ADB is alſo given: Therefore the third angle DAB is given. Wherefore *g* the angle ABD is given by kind; and therefore *h* the reaſon of DA to AB is given. But the reaſon of AB to AC is alſo given: Therefore *i* the reaſon of DA to AC is given, and the angle D is a right angle: Therefore the triangle DAC is given by kind, and therefore the angle ACB is given. But the angle ABC is alſo given: Therefore the third angle BAC is given. Wherefore the triangle ABC is given by kind.



g 40. *prop.*
h 3. *def.*
i 8. *prop.*

PROP. XLV.



If a triangle ABC, hath one angle BAC given, and that the line compounded of the two ſides AB and AC, about the ſaid given angle BAC, hath to the other ſide BC a given reaſon, the triangle ABC is given by kind.

Conſtr. For, let the angle BAC be divided into two equal parts by
a 7. *prop.* the line AD, therefore *a* the angle CAD is given.

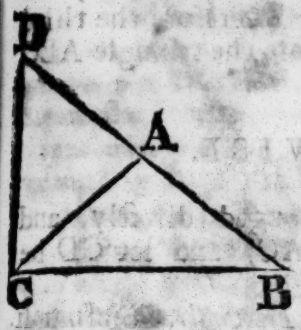
Demonſtr. Seeing that as AB is to AC, ſo
b 3. 6. *b* is BD to CD; by compounding, *c* as the
c 18. 5. line compounded of CAB is to CA, ſo is BC to CD, and by permutation, as the line compounded of CAB is to CB, ſo is CA to CD. But the reaſon of the line compounded of CAB to BC is given; therefore the reaſon of CA to CD is alſo given, and the angle CAD is gi-
d 44. *prop.* ven. Therefore *d* the triangle ACD is given by kind, and therefore the angle C is given. But the angle BAC is alſo given: Therefore
e 40. *prop.* the third angle B is given: Wherefore *e* the triangle ABC is given by kind.

OTHERWISE.

Conſtr. Let BA be prolonged directly unto the point D, in ſuch ſort as that AD may be equal to AC, and let CD be joined.

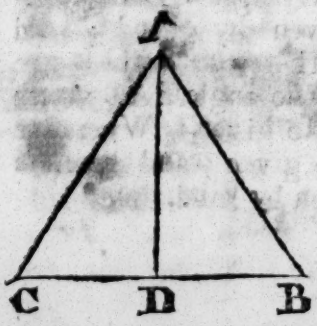
Demonſtr.

Demonſtr. Foraſmuch as the reaſon of the line compounded of CAB to CB is given, and that AD is equal to AC, the reaſon of the whole line BD to BC is given. But the angle ADC is alſo given, for it is the half of the given angle BAC (for that the ſaid angle BAC *f* is equal to the two internal angles ACD and ADC, which are *g* equal to one another, being the ſides AC and AD are equal:) Wherefore the triangle BDC *h* is given by kind, and therefore the angle B is given. But the angle BAC is alſo given: Therefore the remaining angle ACB is given: Wherefore *i* the triangle ABC is given by kind.



f 32. 1.
g 5. 1.
h 44. *prop.*
i 40. *prop.*

PROP. XLVI.



If a triangle ABC hath one angle B given, and that the line CAB compounded of the two ſides, AC and AB, about another angle BAC, hath to the other ſide BC a given reaſon, the triangle ABC is given by kind.

Conſtr. For let the angle BAC be divided into two equal parts, by the line AD.

Demonſtr. Therefore (as hath been ſhewn in the foregoing Propoſition) the compound line CAB is to CB, as AB is to BD. But the reaſon of the ſaid compound line CAB to CB is given: Therefore alſo the reaſon of AB to BD

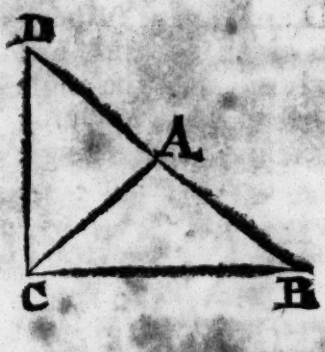
is

is given. But the angle B is alſo given: Therefore the triangle ABD *a* is given by kind; and therefore *b* the angle BAD is given. But the angle BAC is double to that of BAD; and therefore it is alſo given. Therefore the third angle C is given. Wherefore the triangle ABC is given by kind.

a 41. *prop.*
b 2. *def.*

OTHERWISE.

Conſtr. Let BA be prolonged directly, and let AD be put equal to AC, and let CD be joined.

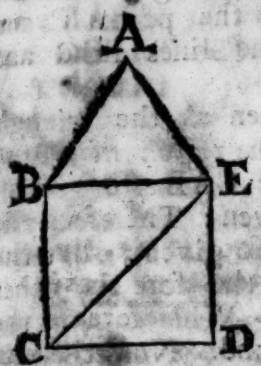


Demonſtr. Foraſmuch as the reaſon of the line compounded of CAB to CB is given, and that AD is equal to AC, the reaſon of BD to BC is given; and the angle B is alſo given: Therefore the triangle CDB *c* is given by kind; and therefore *d* the angle D is given: Therefore the angle BAC which is double to BDC, is alſo given: Wherefore the other angle ACB is given; and therefore the triangle ABC is given by kind.

c 41. *prop.*
d 3. *def.*

PROP.

PROP. XLVII.



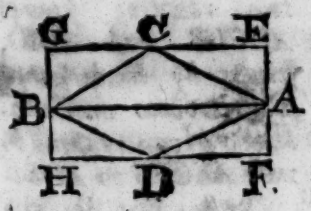
Rectiline figures as ABCDE, given by kind, are divided into triangles given by kind.

Constr. For let the right lines EB and EC be drawn.

Demonstr. Forasmuch as the rectiline figure ABCDE is given by kind, the angle *a* BAE is given, and the reason of the side AB to AE is also given: Therefore *b* the triangle BAE is given by kind. Wherefore the angle ABE is given. But the whole angle ABC is also given: Therefore *c* the remaining angle EBC is given. But the reason of the side AB to the side BE, and also that of AB to BC is given: Therefore *d* the reason of BC to BE is given, and the angle CBE is also given: Therefore *e* the triangle BCE is given by kind. By the same discourse it may be demonstrated that the triangle CDE is given by kind. Therefore the rectiline figures given by kind divide themselves into triangles given by kind.

a 3. *def.*
b 41. *prop.*
c 4. *prop.*
d 8. *prop.*
e 41. *prop.*

PROP. XLVIII.



If on one and the same right line AB, are described triangles as ACB and ABD, given by position, those triangles shall have to one another a given reason, as ACB to ABD,

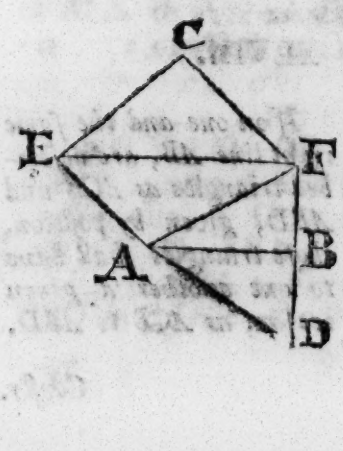
Constr.

Constr. For from the points A and B, let there be drawn at right angles on the line AB, the lines AE and BG, and prolonged unto the points F and H; but by the points C and D, let there be drawn the lines ECG and FDH, parallel to AB.

Demonstr. Forasmuch then as the triangle ABC is given by kind, *a* the reason of CA to BA is given, and the angle CAB also given; but the angle BAE is given: Therefore the remaining angle CAE is also given; but the angle CAE is given; and therefore the other angle ACE is also given. Wherefore *b* the triangle AEC is given by kind. Now the reason of EA to AB *c* is given; (for *d* the reason of EA to AC, and that of AC to AB is given;) and in like manner, the reason of FA to AB is given. Therefore *e* the reason of EA to AF is given; but as AE is to AF, so *f* the parallelogram AH to the parallelogram AG; but ACB is *g* the half of AH, and ADB the half of AG; therefore the reason of the triangle ACB to the triangle ADB is given; for it is the same reason with that of AH to AG *h*; that is to say, of EA to AF, which is given.

a 3. *def.*
b 40. *prop.*
c 8. *prop.*
d 3. *def.*
e 8. *prop.*
f 1. 6.
g 41. 1.
h 15. 5.

PROP. XLIX.



If on one and the same right line AB there be described any two rectiline figures AECFB and ADB, given by kind, they shall have to one another a given reason (to wit) AECFB to ADB.

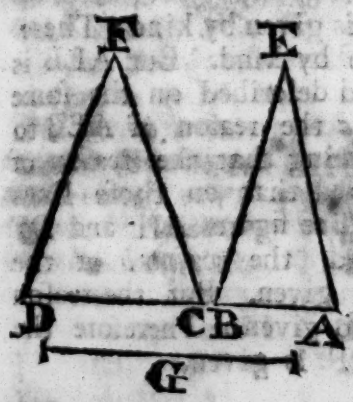
Constr. For let the lines FA and FE be drawn:

drawn: Therefore each of the triangles *a* ABF, AFE, and ECF is given by kind. a 47. *prop.*

Demonstr. Seeing that on one and the same right line EF there are described the triangles ECF and EAF, given by kind; the reason of ECF to EAF *b* is given. Therefore by compounding, *c* the reason of AECF to EAF is given. But the reason of the said EAF to FAB is given, *d* being they are triangles given by kind, described on one and the same right line AF: Therefore *e* the reason of AECF to FAB is given. Wherefore by compounding, *f* the reason of AECFB to FAB is given. But the reason of the same FAB to ABD *g* is given: Therefore *h* the reason of AECFB to ABD is also given.

b 48. *prop.*
c 6. *prop.*
d 48. *prop.*
e 8. *prop.*
f 6. *prop.*
g 48. *prop.*
h 8. *prop.*

PROP. L.



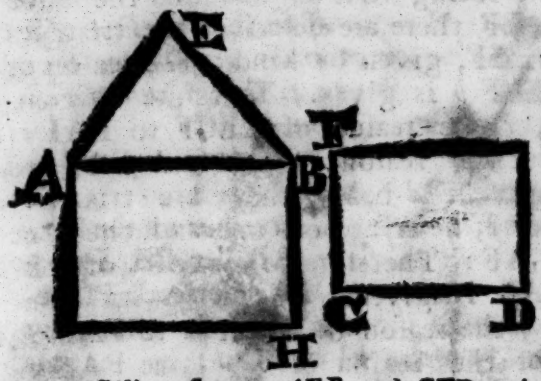
If two right lines AB and CD, have to one another a given reason, and that on those lines there be described rectiline figures AEB and CFD, alike, and alike posited, they will have to one another a given reason.

Demonstr. To the two lines AB and CD, let there be taken a third proportional: Therefore as AB is to CD, so is CD to G. But the reason of AB to CD is given: Therefore the reason of CD to G is also given: Wherefore *a* the reason of AB to G is given. But *b* as AB is to G, so is AEB to CFD: Therefore the reason of the same AEB to CFD is given.

a 8. *prop.*
b *cor.* 19. 20. 6.

PROP.

PROP. LI.



If two right lines AB and CD, have to one another a given reaſon, and that upon them there be deſcribed any rectiline figures AEB and CFD, given by kind, they will have to one another a given reaſon, (to wit, that of AEB to CFD.)

Conſtr. For on AB let the rectangled figure AH be deſcribed alike and alike poſited to DF.

Demonſtr. Now DF is given by kind: Therefore alſo AH is given by kind. But AEB is alſo given by kind, and deſcribed on the ſame line AB: Therefore *a* the reaſon of AEB to AH is given: And ſeeing that the reaſon of AB to CD is given, and that on thoſe lines are deſcribed the rectiline figures AH and DF alike, and alike poſited, the reaſon *b* of the ſaid line AH to DF is given. But the reaſon of AEB to AH is alſo given: Therefore the reaſon *c* of AEB to DF is given.

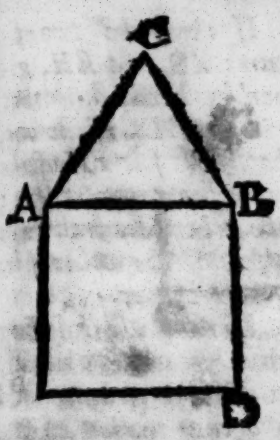
a 49. *prop.*

b 50. *prop.*

c 8. *prop.*

PROP.

PROP. LII.



If on a right line AB, given by magnitude, there be deſcribed a figure ACB, given by kind, that figure ACB is given by magnitude.

Conſtr. For on the ſame line AB, let the ſquare AD be deſcribed. Therefore AD is given by kind * and by magnitude.

Demonſtr. Seeing that on the right line AB, are deſcribed the two rectiline figures ACB and AD, given by kind, *a* the reaſon of ACB to AD is given: Therefore *b* ACB is given by magnitude.

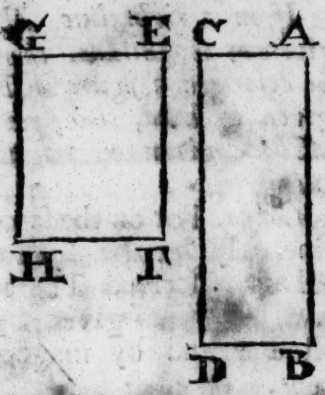
a 29. *prop.*
b 2. *prop.*

Scholium.

* *The antient Interpreter hath noted here that every ſquare is given by kind; for that all the angles thereof are given; being all equal and right angles: But alſo the reaſons of the ſides are given; for thoſe ſides being all equal, their reaſons are alſo equal. Moreover, whenſoever a ſquare is expoſed, a ſquare equal thereto may be exhibited; and therefore the ſquare is given by magnitude, as alſo each ſide thereof.*

PROP.

PROP. LIII.

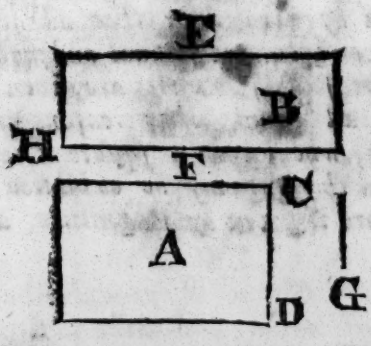


If there be two figures AD and EH, given by kind, and that one side BD, of the one, hath to a side FH of the other, a given reason; the other sides shall have also to the other sides given reasons.

Demonstr. For seeing that the reason of BD to FH is given, and also that *a* of BD to BA, *b* the reason of the said AB to FH is given. But the reason of the same FH to FE *c* is also given: Therefore *d* the reason of AB to EF is given. In like manner also the reasons of the other sides to the other sides are given.

a 3. *def.*
b 8. *prop.*
c 3. *def.*
d 8. *prop.*

PROP. LIV.



If two figures A and B, given by kind, have to one another a given reason, also their sides shall be to one another in a given reason.

Constr. For either the figure A is alike and alike posited to B, or is not: Let it in the first place be alike, and alike posited; and let there be taken the line G, a third proportional to the lines CD and EF.

Demonstr.

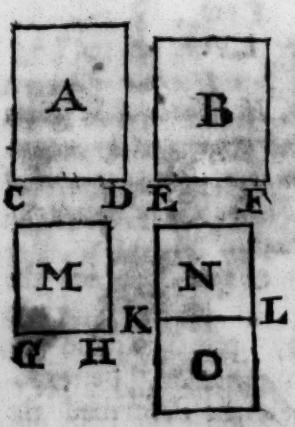
Demonſtr. As CD is to G, *a* ſo is A to B. But the reaſon of A to B is given; therefore alſo the reaſon of CD to G is given. And ſeeing that CD, EF, and G, are proportional, *b* alſo the reaſon of CD to EF is given. But A and B are given by kind: Therefore *c* the other ſides ſhall have given reaſons to the other ſides.

a *cor.* 19, 20. 6.
b 24. *prop.*
c 53. *prop.*

Now let the figure A be not alike to the figure B, and let there be deſcribed on EF the figure EH, alike and alike poſited to A: Therefore the figure EH is given by kind; but the figure B is alſo given by kind: Therefore *d* the reaſon of B to EH is given; and therefore the reaſon of A to the ſame EH *e* is alſo given: But A is alike to EH: Therefore (by what is aboveſaid) the reaſon of CD to EF is given; and in like manner the reaſon of the other ſides to the other ſides is given.

d 49. *prop.*
e 8. *prop.*

OTHERWISE.



Conſtr. Let there be expoſed the given line GH: Now either the figure A is alike to the figure B, or not. Let it in the firſt place be alike, and let it be as CD is to EF, ſo is GH to LK; then on GH and LK let the figures M and N be deſcribed alike, and alike poſited to the ſaid A and B, which figures M and N ſhall be conſequently given by kind.

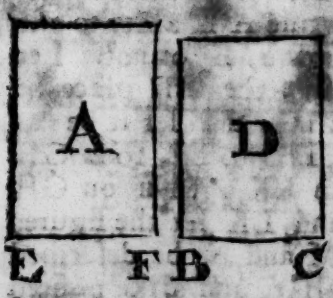
Demonſtr. Therefore ſeeing that as CD is to EF, ſo is GH to LK, and that on thoſe lines CD, EF, GH, and LK, are deſcribed the figures

f 22. 6. A, B, M, and N, alike and alike posited; *f* as A is to B, so is M to N. But the reason of A to B is given: Therefore the reason of M
g 52. *prop.* to N is given, But *g* M is given, considering that it is a figure given by kind, described on a right line given by magnitude; therefore N is also given.

Constr. 2. Now, on LK let the square O be
h *sch.* 52. described: Therefore *h* the figure O is given
prop. by kind.

Demonstr. 2. Wherefore the reason of O to N is given. But N is given: Therefore O
i *sch.* 52. is given; and consequently, *i* also KL. But
prop. GH is given: Therefore *k* the reason of GH to
k 3. *prop.* KL is given. But as GH is to LK, so is CD to EF. Therefore the reason of CD to EF is given; and therefore the figures A and B being
l 53. *prop.* given by kind, *l* the other sides of the same figures shall also have to the other sides given reasons. But if the figures be not alike, the latter part of the demonstration here above must be observed.

PROP. LV.



If a space A be given by kind, and by magnitude, the sides thereof shall be given by magnitude.

Constr. For, let the right line BC given by position and by magnitude, be exposed; and thereon let there be described the space D, alike and alike posited to A; therefore the said space D is given by kind.

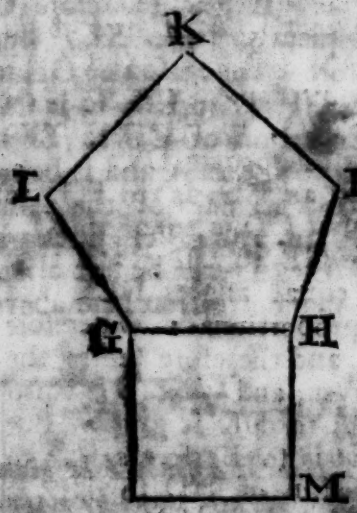
Demonstr. For that it is described on the
a 52 *prop.* line BC, given by magnitude, it is also *a* given by magnitude. But the figure A is also given:

There-

Therefore *b* the reaſon of A to D is given. But thoſe figures A and D are given by kind: Therefore *c* the reaſon of the line EF to the line BC is given. But BC is given: Therefore *d* EF is alſo given. But the reaſon of the ſame EF to FG is given: Therefore *e* FG is given. And by the ſame reaſons it may be demonſtrated that each of the other ſides are given by magnitude.

b 3. *prop.*
c 54. *prop.*
d 3. *def.*
e 2. *prop.*

OTHERWISE.



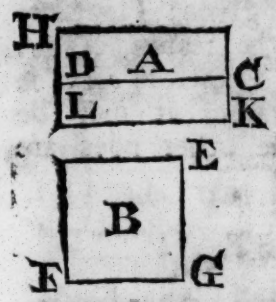
Conſtr. Let the ſpace GHIKL be given by kind and by magnitude: I ſay that the ſides thereof are given by magnitude. For on the right line GH let there be deſcribed the ſquare GM: therefore *f* GM is given by kind.

f *ſch.* 52. *prop.*

Demonſtr. But the ſpace GHIKL is alſo given by kind: Therefore *g* the reaſon of the ſame ſpace GK to GM is given. But GK is given by magnitude: Therefore *h* GM is alſo given by magnitude; and ſeeing that GM is the ſquare of the line GH, *i* that line GH is given by magnitude. Wherefore in like manner, each of the other lines HI, IK, KL, and LG, is given.

g 49. *prop.*
h 2. *prop.*
i *ſch.* 52. *prop.*

PROP. LVI.



If two equiangled parallelograms A and B, have to one another a given reaſon, as one ſide CD of the firſt A, is to one ſide FG, of the ſecond B; ſo the other ſide GE, of the ſecond B, is to that to which DH the other ſide of the firſt A, hath the given reaſon that the parallelogram A hath to the parallelogram B.

Conſtr. For let HD be prolonged directly to L, ſo that as CD is to FG, ſo HD may be to DL; and finiſh the parallelogram DK.

Demonſtr. Seeing that as CD is to FG, ſo
a 34. 1. HD is to DL, and *a* that CD is equal to KL; as LK is to FG, ſo is GE to DL; and thus the ſides about the equal angles DLK and EGF are reciprocally proportional: Wherefore
b 14. 6. *b* DK is equal to B; and therefore ſeeing the reaſon of A to B is given, and that B is equal to DK, the reaſon of A to DK is given. But
c 1. 6. as *c* A is to DK (that is to B) ſo is HD to DL; therefore the reaſon of HD to DL is alſo given: and ſeeing that as CD is to FG, ſo GE is to DL, and that the right line HD hath to DL a given reaſon; to wit, that which the ſpace A hath to the ſpace B; as CD is to FG, ſo GE is to that to which HD hath the given reaſon that the ſpace A hath to the ſpace B, that is to ſay, the reaſon of HD to DL.

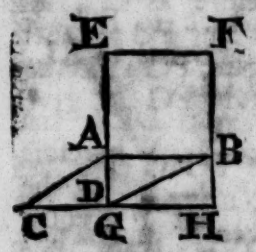
PROP.

PROP. LVII.

If a given ſpace AD be applied to a given right line AB in a given angle CAB, the breadth CA of the application is given.

Conſtr. For on AB, let there be deſcribed the ſquare AF; therefore *a* the ſame AF is given: Let the lines EA, FB, and CD, be prolonged to the points G and H. a *ſch.* 52. *prop.*

Demonſtr. Seeing therefore that each ſpace AD and AF is given, their reaſon is alſo given. But *b* AD is equal to AH: Therefore the reaſon of AF to AH is given: Wherefore the reaſon of EA to AG is given. (for *c* it is the ſame with that of AF to AH.) But EA is equal to AB; therefore the reaſon of AB to AG is given. Now ſeeing that the angle CAB is given, and the angle GAB alſo given, the reſidue CAG is given. But the angle CGA is alſo given, being a right angle: Therefore the remaining angle ACG is given. Wherefore the triangle *d* CAG is given by kind. Therefore the reaſon of CA to AG is given. But the reaſon of AB to the ſame AG is alſo given: Therefore the reaſon of CA to AB is given; and the ſaid AB is given: Wherefore CA is alſo given.

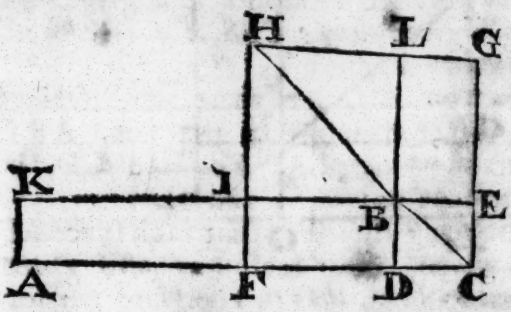


b 36. 1.

c 1. 6.

d 40. *prop.*

PROP. LVIII.



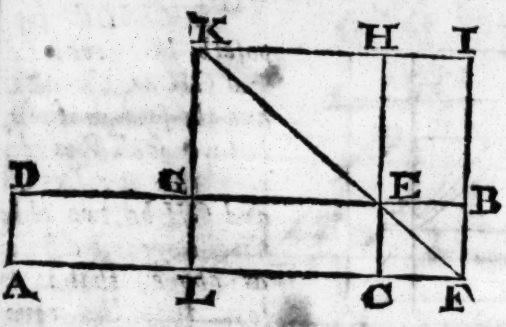
If a given ſpace AB, be applied to a given right line AC, wanting by a figure DE, given by kind, the breadths of the defects are given.

Conſtr. For let AC be divided in two equal parts in the point F: Therefore as well AF as FC is given. On the ſaid line FC let there be deſcribed the rectangled figure FG alike and alike poſited to DE. Therefore FG is given by kind.

Demonſtr. Seeing the figure FG is deſcribed on the right line FC given by magnitude, the
a 52. *prop.* ſaid rectiline FG is *a* alſo given by magnitude.
b 36. 1. But FG is equal to AB and IL; (for *b* AI
c 43. 1. and FE being equal, and *c* FB and BG alſo equal, the Gnomon ICL is equal to AB; and therefore their added figure IL, common to both, FG ſhall be equal to AB and IL:) Therefore the figures AB and IL together are given by magnitude. But AB is given by
d 4. *prop.* magnitude: Therefore *d* the remaining figure IL is alſo given by magnitude. But it is alſo
e 24. 6. given by kind, ſeeing it is *e* alike to DE:
f 55. *prop.* Therefore *f* the ſides of the ſame IL are given: Wherefore IB is given; and ſeeing
g 34. 1. that it is equal *g* to FD, the ſame FD is al-
h 4. *prop.* ſo given. But FC is given, therefore the re-
i 3. *def.* mainder DC *h* is given; and *i* in a given
k 2. *prop.* reaſon to BD, and therefore *k* BD is given.

PROP.

PROP. LIX.



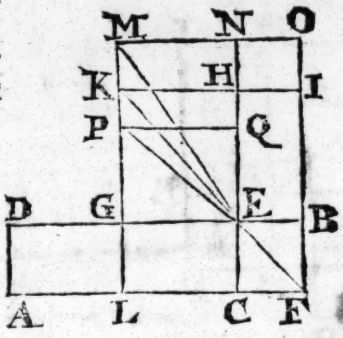
If a given ſpace AB be applied according to a given right line AC, exceeding it by a figure CB given by kind, the breadths of the exceſſes CE and CF are given.

Conſtr. For DE being divided into two equal parts in G, let there be deſcribed on GE the rectiline figure GH, alike and alike poſited to CB.

Demonſtr. Now ſeeing that CB is alike to GH, thoſe figures CB and GH * are about one and the ſame diameter, and GH is given by kind, as is CB. But it is deſcribed on the given line GE: Therefore *a* the ſame GH is alſo given by magnitude. But AB is given: Therefore AB and GH are given by magnitude. Now thoſe figures AB and GH, are equal to LI, (for AG, LE, and EI, being equal, the Gnomon GFH is equal to AB; and therefore adding GH common to both, LI ſhall be equal to AB and GH;) therefore LI is given by magnitude; but it is alſo given by kind, being it is *b* alike to CB. Therefore *c* the ſides of the ſaid LI are given, ſeeing it is equal to GE: Therefore *d* the remainder CF is given, and in a given reaſon *e* to CE. Wherefore *f* CE is given.

a 32. *prop.*
b 24. 6.
c 55 *prop.*
d 4. *prop.*
e 3. *def.*
f 2 *prop.*

Scholium.



* *EUCLIDE ſuppoſeth here that CB and GH are about one and the ſame diameter, but we ſhall thus demonſtrate it: Let CB and GH be two alike parallelograms diſpoſed as above, that is to ſay, that the equal angles join together in E, the ſide CE meets directly with his homologal ſide EH, and the ſide BE, his correſpondent ſide EG; and let the diameter FE be drawn, I ſay that the ſaid diameter FE prolonged, will paſs by the point K; that is to ſay, the parallelograms GH and CB, conſiſt about one and the ſame diameter. For if it be denied, the diameter EF being produced, will paſs above the point K, or below it. Let it in the firſt place paſs above it, and let it cut GK, prolonged in the point M, and by the point M, let there be drawn MN, parallel to KH, which ſhall meet EH, prolonged in the point N, and FB in O.*

Demonſtr. Foraſmuch as the parallelograms GN and CB are with the parallelogram LO about one and the ſame diameter, they are *g* alike to one another. Wherefore as FC is to CE, ſo is EG to GM. In like manner, ſeeing the parallelograms CB and GH are alike, as FC is to CE, ſo is EG to GK: Therefore *h* as EG is to GM, ſo is EG to GK. Wherefore *i* GM and GK are equal, a part to the whole, which is abſurd: By the ſame reaſons it may be demonſtrated; that the diameter prolonged will not fall below the point K: There-

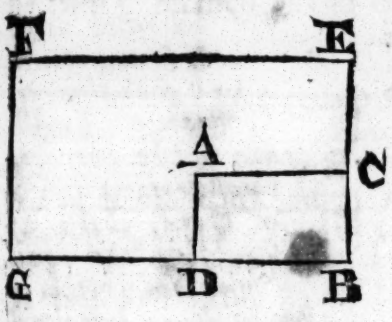
g 24. 6.

h 11. 5.

i 9. 5.

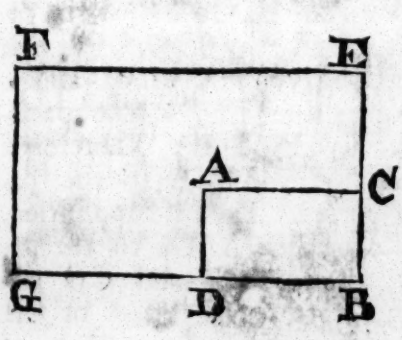
Therefore the parallelograms CB and GE conſiſt about one and the ſame diameter.

PROP. LX.



If a parallelogram AB, given by kind and by magnitude, be augmented or diminiſhed by a Gnomon CFD, the breadths of the Gnomon (conſiſting of the lines CE and DG) are given.

Demonſtr. For ſeeing that AB is given, and the Gnomon CFD alſo given, the whole parallelogram BF is given: But it is alſo given by kind, ſeeing it is alike to BA: Therefore *a* the ſides of the ſame BF are given; and a 55. *prop.* therefore each of the lines BE and BG is given. But each of the lines BC and BD is given; therefore each of the remaining lines CE and DG is alſo given.

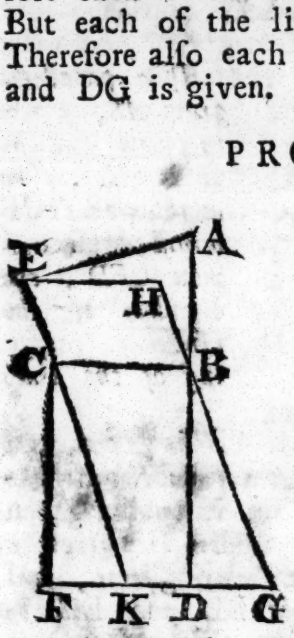


Conſtr. Now let the parallelogram BF, given by kind and by magnitude, be diminiſhed by the given Gnomon CFD: I ſay that each of the lines CE and DG is given.

Demonſtr. 2. For ſeeing that BF is given, and the Gnomon CFD given, the remaining figure AB is alſo given. But it is alſo given by

by kind, ſeeing it is alike to BF: Therefore
b 55. *prop.* *b* the ſides of the ſaid AB are given, and therefore each of the lines CB and BD is given. But each of the lines BE and BG is given: Therefore alſo each of the remaining lines CE and DG is given.

PROP. LXI.



If to one ſide of a figure ABCE, given by kind, there be applied a ſpace parallelogram CD, in a given angle BCF, and that the given figure AC hath to the parallelogram CD a given reaſon, the parallelogram CD is given by kind.

Conſtr. For by the point B, let BH be drawn parallel to CE, and by the point E let EH be drawn parallel to CB, and let EC and HB be prolonged to the points K and G.

Demonſtr. Foraſmuch as the angle BCE is
a 3. *def.* given, and the reaſon of EC to CB, *a* the parallelogram CH is given * by kind. But the figure ABCE is alſo given by kind, and is deſcribed on the ſame line BC, as the parallelogram CH given by kind is: Therefore *b*
b 49. *prop.* the reaſon of the figure ABCE to the parallelogram CH is given. But by ſuppoſition, the reaſon of the ſaid figure ABCE to the pa-
c 36. 1. rallelogram CD is alſo given; and CD is *c*
d 8. *prop.* equal to CG: Therefore *d* the reaſon of CH to CG is given. Wherefore the reaſon of
e 1. 6. the line EC to the line CK is given; (for *e* as CH is to CG, ſo is EC to CK.) But the reaſon of EC to CB is alſo given: Therefore
f 8. *prop.* *f* the reaſon of the ſaid CB to CK is given.

And

And ſeeing that the angle ECB is given, alſo the following angle BCK *g* is given. But the angle BCF is propoſed given; and therefore the remaining angle FCK is given. Alſo the angle CKF is given, for that *h* it is equal to the angle BCK: Therefore the other angle CFK is given: Wherefore *i* the triangle FCK is given by kind; and therefore the reaſon of FC to CK is given. But the reaſon of CB to the ſame CK is alſo given: Therefore *k* the reaſon of FC to CB is given; and the angle BCF is alſo given. Wherefore the parallelogram CD is given by kind.

g 13. 1. 4. *prop.*
h 29. 1.
i 40. *prop.*
k 8. *prop.*

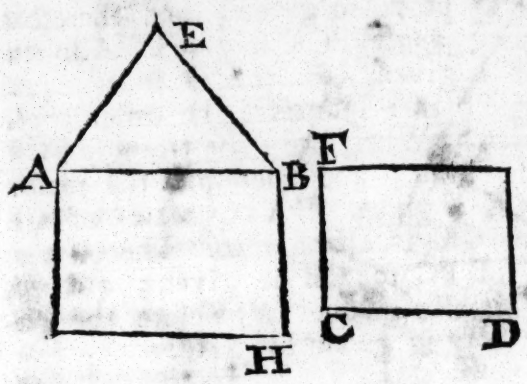
Scholium.

* *Altho' it be manifeſt that a parallelogram that hath one angle given, and the reaſon of the ſides about the ſame angle alſo given, is given by kind, as* Euclide *doth here declare, ſo it is notwithſtanding that the antient Interpreter doth thus demonſtrate it.*

Seeing that in the parallelogram CH the angle ECB is given, the angle CEH is alſo given; for the right line EC falling on the parallels EH and CB, doth make the two internal angles on the ſame part equal to two right angles. And therefore ſeeing that the angle ECB is given, the other angles are given; and ſeeing that the reaſon of EC to CB is given, and that BH is equal to CE, and EH to BC, the reaſon of the ſides to one another is alſo given.

PROP.

PROP. LXII.



If two right lines AB and CD, have to one another a given reaſon, and that on one of them AB, there be deſcribed a figure AEB, given by kind; but on the other CD, a ſpace parallelogram DF, in a given angle DCF, and that the figure AEB hath to the parallelogram DF a given reaſon, the parallelogram DF is given by kind.

Conſtr. For on the line AB let there be deſcribed the parallelogram AH, alike and alike poſited to DF.

Demonſtr. Seeing then that the reaſon of AB to CD is given, and that on thoſe lines are deſcribed the rectiline figures AH and FD, alike and alike poſited, *a* the reaſon of AH to FD is given. But the reaſon of FD to AEB is alſo given: Therefore *b* the reaſon of AH to AEB is given. But the angle ABH is alſo given, being equal to the angle FCD, and ſo the figure AEB is given by kind; and to AB one of the ſides thereof, the parallelogram AH is applied in a given angle ABH, and the reaſon of the ſaid figure AEB to the ſaid parallelogram AH is given: Therefore *c* the parallelogram AH is given by kind; and therefore FD which is alike thereto, is alſo given by kind.

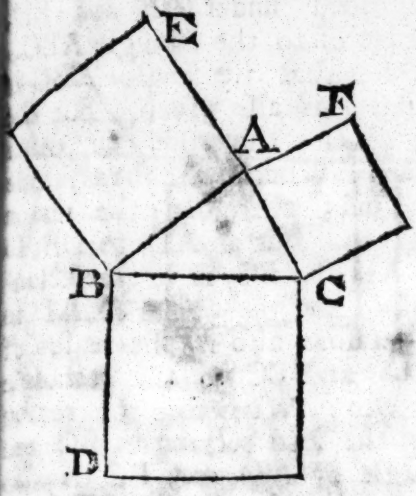
a 50. *prop.*

b 8. *prop.*

c 61. *prop.*

PROP.

PROP. LXIII.

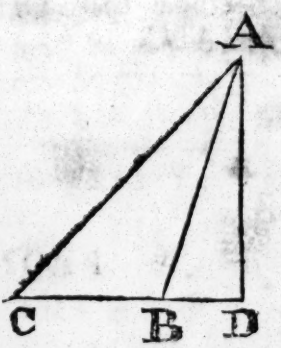


If a triangle ABC be given by kind, the ſquare BE, CD, and CF, which is deſcribed on each of the ſides, ſhall have a given reaſon to the triangle ABC.

Demonſtr. For ſeeing that on one and the ſame right line BC, there are deſcribed the two rectiline figures ABC and CD, given by kind, *a* the reaſon of the ſame ABC to CD is given; and therefore the reaſon of the ſquares BE and CF, to the triangle ABC, is alſo given.

a 49. *prop.*

PROP. LXIV.



If a triangle ABC, hath an obtuſe angle ABC given, that ſpace of which the ſide AC ſubtending the obtuſe angle ABC, is more in power than the ſides AB and BC, that comprehend the ſaid angle, ſhall have a given reaſon to the triangle ABC.

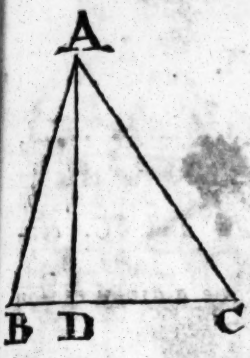
Conſtr. Let the line CB be prolonged directly, and from the point A let the perpendicular AD be drawn: I ſay

ſay that the ſpace of which the ſquare of the line AC doth exceed the ſquares of the lines
a 12. 2. AB and BC, that is to ſay, *a* the double of the rectangle contained under CB and BD, ſhall have a given reaſon to the triangle ABC.

Demonſtr. For ſeeing that the angle ABC is given, the angle ABD is alſo given. But the angle ADB is alſo given; therefore the other
b 40. *prop.* angle BAD is given: Wherefore *b* the triangle
c 3. *def.* ABD is given by kind; therefore *c* the reaſon of AD to DB is given. But as AD to DB, ſo
d 1. 6. *d* the rectangle of AD and BC is to the rectangle of BC and BD. But the reaſon of AD to BD is given: Therefore alſo is the reaſon of the rectangle of AD and BC to the rectangle of BC and BD given: Wherefore the reaſon of the double of the ſaid rectangle BC and BD to the rectangle of AD and BC is alſo given. But the ſaid rectangle of AD and BC hath alſo a given reaſon to the triangle ABC (to wit, double reaſon; for the rectangle
e 41. 1. is *e* double to the triangle) therefore the reaſon of the double of the rectangle of BC and
f 8. *prop.* BD *f* to the triangle ABC is given. But the ſame double of the rectangle of CB and BD is that ſpace of which the ſquare of the line AC doth exceed the ſquares of the lines AB and BC: Therefore the ſame ſpace hath a given reaſon to the triangle ABC.

PROP.

PROP. LXV.



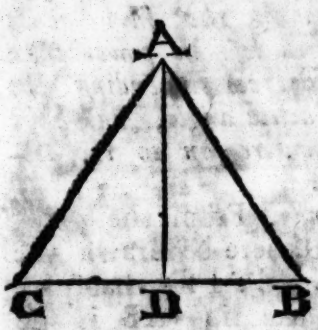
If a triangle ABC, hath one acute angle ACB given, that ſpace, of which the ſide ſubtending the ſaid acute angle is leſs in power than the ſides comprehending the ſame acute angle, ſhall have a given reaſon to the triangle.

Conſtr. From the point A let there be drawn the line AD, perpendicular to BC: I ſay that ſpace of which the ſquare of the line AB is leſs than the ſquares of the lines AC and CB, that is to ſay, *a* the double of the rectangle of BC and CD, hath a given reaſon to the triangle ABC. a 13. 2.

Demonſtr. For ſeeing that the angle C is given, and the angle ADC alſo given, the other angle DAC is given: Wherefore the triangle *b* ADC is given by kind; and therefore the reaſon of AD to DC is given, and conſequently alſo *c* that of the rectangle of BC and CD to the rectangle of BC and AD: Therefore the reaſon of the double of the rectangle of BC and CD to the rectangle of BC and AD is given. But the reaſon of the ſame rectangle of BC and AD to the triangle ABC is given (for *d* the rectangle is double to the triangle:) Therefore *e* the reaſon of the double of the rectangle of BC and CD to the triangle ABC is given. And ſeeing that the ſame double of the rectangle of BC and CD is that whereof the ſquare of the line AB is leſs than the ſquares of the lines AC and BC, that ſpace of which the ſquare of the line AB b 40. *prop.* c 1. 6. d 41. 1. e 8. *prop.*

AB is leſs than the ſquares of the lines AC and BC, ſhall have a given reaſon to the triangle ABC.

PROP. LXVI.



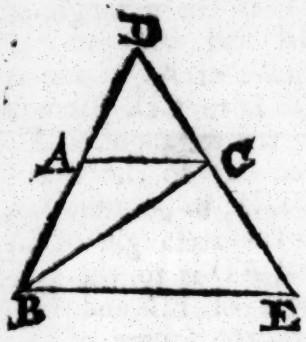
If a triangle ACB, hath one angle B given, the rectangle made of the lines AB and BC, containing the ſame angle, ſhall have a given reaſon to the triangle.

Conſtr. For from the point A let AD be drawn perpendicular to CB.

Demonſtr. Therefore ſeeing that the angle B is given, and alſo the angle ADB; the other angle BAD is likewiſe given. Wherefore the
a 40. *prop.* triangle ADB *a* is given by kind; and conſequently the reaſon of AB to AD is given.
b 1. 6. But as AB is to AD, *b* ſo the rectangle of AC and CB is to the rectangle of CB and AD: Therefore the reaſon of the rectangle of AC and CB to the rectangle of CB and AD is given. But the reaſon of the ſaid rectangle of CB and AD to the triangle ACB is alſo given; (for that it is double reaſon, the
c 41. *prop.* rectangle being double *c* to the triangle:)
d 8. *prop.* Therefore *d* the reaſon of the rectangle of AC and CB to the triangle ABD is given.

PROP

PROP. LXVII.



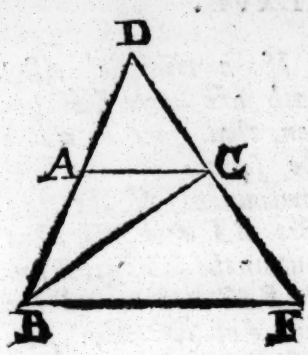
If a triangle ABC hath one angle BAC given, that ſpace by which the ſquare of the line compounded of the two ſides BA and AC, that contain the ſame given angle BAC doth exceed the ſquare of the other ſide, it ſhall have a given reaſon to the triangle ABC.

Conſtr. For let BA be prolonged in ſuch ſort as that AD may be equal to AC, then having drawn DCE infinitely, from the point B let BE be drawn parallel to AC, meeting the ſaid DE in the point E.

Demonſtr. Foraſmuch as AD is equal to AC, *a* DB is equal to BE; (for the two triangles ADC and BDE are alike) and from the top B is drawn to the baſe DE, the right line BC: Therefore * the rectangle of DC and CE, with the ſquare of BC, is equal to the ſquare of BD; but the ſame BD is compounded of BA and AC; therefore the ſquare of the compound of AB and AC is greater than the ſquare of BC, of the rectangle of DC and CE. a 4. 6. & 14. 5.

Now I ſay that the rectangle of DC and CE hath a given reaſon to the triangle ABC: Foraſmuch as the angle BAC is given, the angle DAC is alſo given. But each of the angles ADC and ACD is given, it being the half of the angle BAC which is given. Therefore *b* the triangle ADC is given by kind; and therefore the reaſon of DA to DC is given. Therefore *c* the reaſon of the ſquare of the ſaid DA to the ſquare of DC is alſo b 40. *prop.* c 50. *prop.*

d 2. 6. given. And seeing that as BA is to AD, d so is EC to CD, and also as BA is to AD,



e 1. 6. e so is the rectangle of BA and AD to the square of AD; and as

f 1. 6. EC is to CD, f so also is the rectangle of EC and CD to the square of CD; by permutation, as the rectangle of BA and AD is to the rectangle of EC and CD, so is the square of AD to the square of DC. But the reason of the said square of AD to the square of DC is given: Therefore the reason of the rectangle of BA and AD to the rectangle of EC and CD is also given. But AD is equal to AC: Therefore the reason of the rectangle of BA and AC to the rectangle of EC and CD is given. But the reason of the rectangle of BA and AC to the triangle

g 6. *prop.* ABC *g* is given, because the angle BAC is

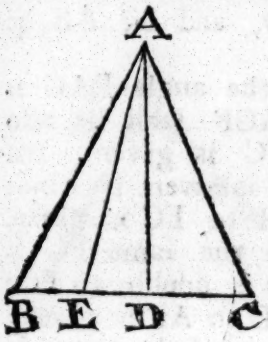
h 8. *prop.* given: Therefore *h* the reason of the rectangle EC and CD to the triangle ABC is given. But the rectangle of EC and CD is that whereof the square of the line compounded of BA and AC is greater than the square of BC: Therefore that space to which the square of the line compounded of BA and AC is greater than the square of BC, shall have a given reason to the triangle ABC.

Scholium.

* *EUCLIDE supposeth in this place that when in an Isosceles triangle a right line is drawn from the top to the base, the square of that line, with the rectangle contain'd under the segments of the*

the baſes, is equal to the ſquare of either of the other legs, which the antient interpreter doth thus demonſtrate.

Conſtr. Let ABC be an Iſoſceles triangle, whoſe legs are AB and AC; and from the top A let AD be drawn to the baſe BC: I ſay that the ſquare of AD with the rectangle of BD and DC, is equal to the ſquare of either of the legs AB or AC.



Demonſtr. Now the line AD is perpendicular to BD, or not: Let it in the firſt place be perpendicular: Therefore it will cut the baſe BC into two equal parts in the point D; and therefore the rectangle contained under BD and DC is equal to the ſquare of the ſaid BD, and adding to them the common ſquare of AD, the rectangle of BD and DC with the ſquare of AD, ſhall be equal to the ſquares of DB and AD. But to thoſe ſquares of AD and DB *i* the ſquare of AB is equal: Therefore the ſquare of AB is equal to the rectangle of BD and DC, and the ſquare of AD together. i 47. 1.

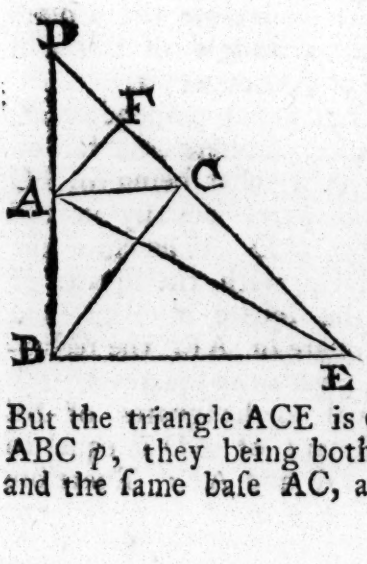
Now ſuppoſe AD not to be perpendicular, but that from the point A there doth fall on BC the perpendicular AE, that being ſo, BC ſhall be cut into two parts equally in the point E, and unequally in D. Wherefore the rectangle of BD and DC, with the ſquare of DE, *k* is equal to the ſquare of BE; and adding the common ſquare of AE, the rectangle of BD and DC, with the ſquares of DE and AE, ſhall be equal to the ſquares of BE and AE. But *l* the ſquare of AD is equal to the two ſquares of DE and AE: Therefore k 5. 2. l 47. 1.

the rectangle of BD and DC, with the ſquare of AD is equal to the ſquares of BE and AE. But to theſe ſquares of BE and AE the ſquare of AB is equal: Therefore the ſquare of AD, with the rectangle of BD and DC, is equal to the ſquare of AB.

OTHERWISE.

Conſtr. Having done, as in the foregoing Demonſtration, from the point A, let AF be drawn perpendicular to CD, and let AE be drawn.

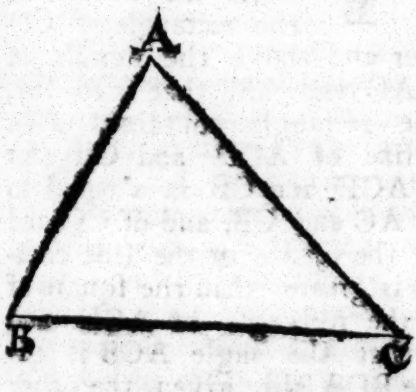
Demonſtr. Foraſmuch as the angle BAC is given, the half thereof ACF ſhall be alſo given. But the angle AFC is given; and therefore the triangle AFC is given by kind: Therefore the reaſon of AF to FC is given. But the reaſon of CD to the ſame FC is alſo given, ſeeing that CD is double to FC:
m 8. *prop.* Therefore *m* the reaſon of CD to AF is given; and therefore alſo the reaſon of the rectangle of CD and EC, to the rectangle of AF and
n 1. 6. EC, is given; (for it is the ſame reaſon *n* as that of CD to AF.)



o 41. 1. But the reaſon of the rectangle of AF and FC to the triangle ACE is given; ſeeing it is double *o* to the ſame triangle. Therefore the reaſon of the rectangle of CD and CE to the triangle ACE is alſo given.
p 37. 1. But the triangle ACE is equal to the triangle ABC *p*, they being both conſtituted on one and the ſame baſe AC, and between the ſame parallels

parallels AC and BE: Therefore *q* the reaſon of the rectangle of CE and CD to the triangle ABC is given. But the ſaid rectangle of CE and CD is the ſpace by which the ſquare of the line compounded of AB and AC, is greater than the ſquare of BC: Therefore that ſpace by which the ſquare of the line compounded of AB and AC is greater than the ſquare of BC, hath a given reaſon to the triangle ABC. q 8. *prop.*

OTHERWISE.

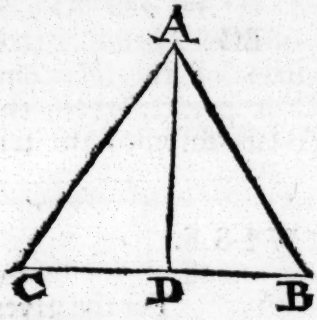


For the given angle A is either a right, acute, or obtuſe angle: Let it in the firſt place be ſuppoſed a right angle: Therefore the ſquare of the line compounded of BAC, is greater than the ſquare of BC, by twice the rectangle of BA and AC; (ſeeing that *r* the ſquare of BC is equal to the ſquares of BA and AC; and the ſquare of the line compounded of BAC *s* is equal to thoſe two ſquares of BA and AC, and twice the rectangle of the ſaid BA and AC:) Wherefore the reaſon of double the rectangle of BA and AC to the triangle ABC is given. r 47. 1. s 5. 2.

Conſtr. Now let the angle C be ſuppoſed acute, and from the point A let there be drawn on CB the perpendicular AD.

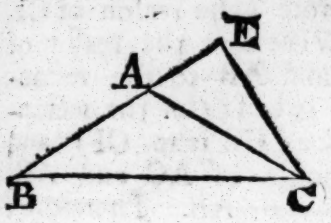
Demonſtr. Foraſmuch as the triangle CAB is an Oxigonium triangle, and the perpendicular AD being drawn, the ſquare of CA and CB

t 13. 5. are equal *t* to the ſquare of AB with twice the rectangle of CB and CD; adding therefore the common double rectangle of CA and CB, the ſquares of CA and CB, with the double rectangle of the ſaid CA and CB, that is to ſay, *u* the alone ſquare of the line compounded of ACB, are equal to the ſquare of AB, with the double of the rectangle of CD and CB, and over and above the double of the rectangle of AC and CB, that is to ſay, the double of the rectangle contained under the compound line of ACD and CB (for x 1. 2. the rectangle of ACD and CB is *x* equal to the rectangles of AC and CB, and of CD and CB:) Therefore the ſquare of the line compounded of ACB is greater than the ſquare of AC, by double the rectangle of ACD and CB. And ſeeing that the angle ACB is given, and the angle BDA alſo given, the other y 40 *prop.* angle CAD is given: Therefore *y* the triangle CAD is given by kind, and therefore the reaſon of CD to CA is given, and by conſequence the reaſon of the line compounded z 6. *prop.* of ACD to CA *z* is alſo given. Wherefore the reaſon of the rectangle of thoſe lines a 1. 6. compounded of ACD and CB *a* to the rectangle of AC and CB is alſo given. But the reaſon of the ſaid rectangle of AC and CB b 66. *prop.* to the triangle CAB *b* is given, ſeeing the angle C is given; therefore the reaſon of double the rectangle of the line compounded of ACD and CB to the triangle CAB is given.



u 4. 2.

Laſtly,



Laſtly, Let the angle BAC be ſuppoſed to be obtuſe, and having prolonged BA from the point C, let the perpendicular CE be drawn on the ſaid line BA prolonged; and let AF be propoſed to be equal to AE.

Demonſtr. Foraſmuch as the angle BAC is obtuſe, and the perpendicular CE being drawn, the ſquares of AB and AC, and the double of the rectangle under BA and AE, or AF, are all alike equal *c* to the ſquare of BC, and adding the common double rectangle of BA and AC, the ſquares of the ſaid AB and AC, with the double of the rectangle of the ſame AB and AC, that is to ſay, *d* the ſquare of the line compounded of BAC, and the double of the rectangle of BA and AF are together equal to the ſquare of BC, with the double of the rectangle of BA and AC. Let the common double of the rectangle of BA and AF be taken away, and there will remain the ſquare of the line compounded of BAC, equal to the ſquare of BC, with the rectangle of AB and CF; (for the rectangle of AB and AC is equal *e* to the two rectangles of AB and AE, and of AB and CF:) Therefore the ſquare of the line compounded of BAC is greater than the ſquare of BC by the double of the rectangle of AB and CF. And foraſmuch as the angle BAC is given, the angle CAE *f* is given. But the angle AEC is alſo given; therefore the other angle ACE is given: Wherefore *g* the triangle ACE is given by kind, and therefore the reaſon of CA to AE, that is to ſay, to AF is given. Therefore *h* the reaſon of the ſaid CA to FG is

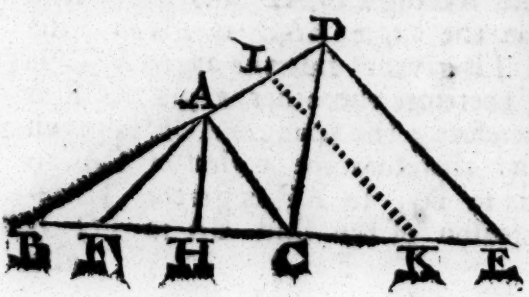
c 12. 2.
d 4. 2.
e 1. 2.
f 13. 11.
g 40. *prop.*
h 5. *prop.*

alſo given. But the reaſon of the ſame CA
i 8. *prop.* to CE is given; therefore *i* the reaſon of CE to CF is alſo given. Wherefore the reaſon of the rectangle of EC and AB to the rectangle of FC and AB is given; (for the rectan-
k 2. 6. gle is to the rectangle *k* as CE is to CF) and alſo that of the rectangle of AC and AB,
l 8. *prop.* to the rectangle of EC and AB. Therefore *l* the reaſon of the rectangle of FC and AB to the rectangle of AC and AB is given. But the reaſon of the rectangle of AC and AB
m 66. *prop.* to the triangle ABC *m* is given: Therefore alſo the reaſon of the double of the rectangle of FC and AB, to the triangle ABC is given. But the ſame double of the rectangle of FC and AB is that, whereof the ſquare of the line compounded of BAC is greater than the ſquare of BC, whereof that ſpace of which the ſquare of the line compounded of BAC is greater than the ſquare of BC, hath a given reaſon to the triangle ABC.

OTHERWISE.

Conſtr. Let the line BA be prolonged to the point D, in ſuch ſort as AD may be equal to AC, and let CD be drawn.

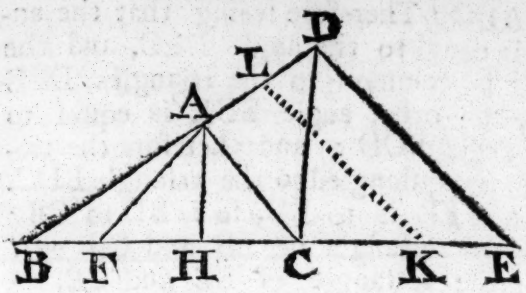
Demonſtr. Foraſmuch as the angle BAC is given, each of the angles ADC and ACD, which is the half thereof ſhall be alſo given; and therefore the other angle DAC is alſo
n 40. *prop.* given: Therefore *n* the triangle ACD is given by kind. Wherefore the reaſon of AC to CD is given. And foraſmuch as the angle ADC is given: Let each of the angles DEC and AFC be made † equal to the



the ſaid ADC: Therefore ſeeing that the angle BDC is equal to the angle DEC, and the angle DBE is common to the triangles DBE and DBC, the other angle BDE is equal to the other angle BCD; and therefore the triangle BDE is equiangled to the triangle BDC. Therefore *o* as EB is to BD, ſo is BD to CB: Wherefore the rectangle of EB and CB, that is to ſay, *p* the rectangle of EC and CB, *q* with the ſquare of CB is equal, *r* to the ſquare of BD, that is to ſay, to the ſquare of the line compounded of BAC; for AD is equal to AC; and therefore the rectangle of EC and CB, with the ſquare of CB, that is to ſay, the ſquare of the line compounded of BAC is greater than the ſquare of the rectangle of BC and CE: I ſay therefore that the reaſon of the ſaid rectangle of BC and CE to the triangle ABC is given. Foraſmuch as the angle BDE is equal to the angle BCD, and the angle ADC equal to the angle ACD, the other angle CDE is equal to the other angle ACB: But the angle DEC is alſo equal to the angle AFC; therefore the remaining angle CAF is equal to the remaining angle DCE. Wherefore the triangle AFC is equiangled to the triangle DCE; and therefore *s* as CA is to AF, ſo is CD to CE; and by permutation, as AC is to CD, ſo is AF to CE. But the reaſon of AC to CD is given: Therefore alſo the reaſon of AF to CE is given. From the point A let AH be drawn perpendicular to BC: Foraſmuch as the angle AFC is given, and the angle AHF alſo given, the third angle HAF is given: Wherefore *t* the triangle AHF is

o 4. 6.
p 5. 2.
q 5. 2.
r 17. 6.
s 4. 6.
t 40. *prop.*

given



given by kind; and by conſequence the reaſon of AF to AH is given.

But the reaſon of AF to CE is alſo given:
u 8. *prop.* Therefore *u* the reaſon of AH to CE is given; and therefore the reaſon of the rectangle
x 1. 6. of AH and BC *x* to the rectangle of BC and CE is alſo given. But the reaſon of the rectangle of AH and BC to the triangle ABC is
y 41. 1. likewiſe given; (for the rectangle *y* is double to the triangle) and the rectangle of BC and CE is that whereof the ſquare of the line compounded of BAC is greater than the ſquare of BC. Therefore that ſpace of which the ſquare of the line compounded of BAC is greater than the ſquare of BC by a given reaſon to the triangle ADC.

Scholium.

† *The antient Interpreter pretending to ſhew the conſtruction of the angle DEB equal to the angle ADC, ſaith that on the line BD and in the point D, the angle BDE ought to be made equal to the angle BCD, and that the right lines BC and DE be drawn until they interſect in E, in ſuch ſort as he ſuppoſeth the angle BCD, to be given, but it is not.*

The ſame Interpreter afterward ſhews how there may univerſally from a given point, be drawn a right line given by poſition to a right line, making an angle equal to a given angle. But we will alſo reject this way, ſeeing we have elſewhere ſhewn

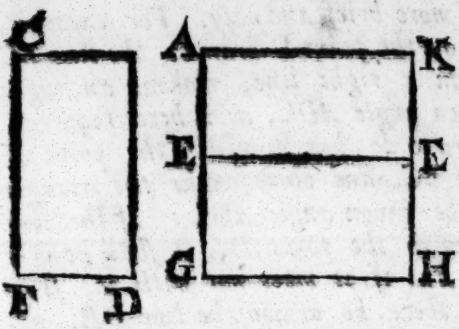
ſhewn another more brief and eaſy. For example, if we would from the point D draw to the line BC given by poſition a right line, making an angle equal to a given angle ADC, as is here required, we have no more to do but to aſſume the point K in the ſaid line BC, and there make the triangle CKL equal to the given angle ADC: If the line KL doth meet with the point D, it ſhall be the line required. But if it meet not with it, from the point D let there be drawn the line DE, parallel to the ſaid KL, cutting BC prolonged in E, and the angle DEC ſhall be equal to the given angle ADC, for on the two parallel lines LK and DE, there doth fall the line BE; and therefore the angle DEC z is equal to the angle LKC, which z 29. 1. *hath been made equal to the given angle ADC; and by conſequence the ſame angle DEC is alſo equal to ADC.*

PROP. LXVIII.

If two parallelograms AB and CD, have to one another a given reaſon, and that a ſide hath alſo a given reaſon to a ſide, the other ſide ſhall have likewiſe a given reaſon to the other ſide.

Conſtr. Let the reaſon of BE to FD be given: I ſay the reaſon of AE to FC is alſo given. For to the right line EB let there be applied the parallelogram FH, equal to the parallelogram CD, and conſtituted in ſuch ſort as AE and EG may make one right line: † Therefore KB and BH will alſo make one right line.

Demonſtr.



a 1. 6.

Demonſtr. Foraſmuch as the reaſon of AB to CD is given, and that EH is equal to the ſaid CD; the reaſon of AB to EH is given; and therefore the reaſon of AE to EG is alſo given. Seeing therefore that EH is equal and equiangled to CD, as *b* EB is to FD, ſo is FC to EG. But the reaſon of EB to FD is given: Therefore alſo the reaſon of FC to EG is given. But the reaſon of AE to the ſame EG is alſo given: Therefore the reaſon of AE to FC is given.

b 14. 6.

Scholium.

† *EUCLIDE having poſited AE and EG directly in one right line, preſently concludeth that KB and BH ſhall alſo make a right line; but we ſhall demonſtrate it thus. Seeing the lines AE and EG are poſited directly, the angles AEB and BEG c are equal to two right angles; and ſeeing that AB is a parallelogram, the lines AK and EH are parallels, on which the line AE doth fall; and therefore the two internal angles A and BEA d are alſo equal to two right angles, and taking away the common angle BEA, there will remain the angle A, equal to the angle BEG; and conſequently their oppoſite angles EBK and H are alſo equal to one another: Again, ſeeing that BG is a parallelogram, the two lines BE and HG are parallels, on which BH doth fall; and therefore the two internal angles H and EBH d are equal to two right angles. But it hath been demonſtrated that*

c 13. 1.

d 29. 1.

that H is equal to EBK: Therefore the two angles EBK and EBH are alſo equal to two right angles; and therefore e *the two lines KB and BH do meet directly according to EUCLIDE.* e 14. 1.

OTHERWISE.

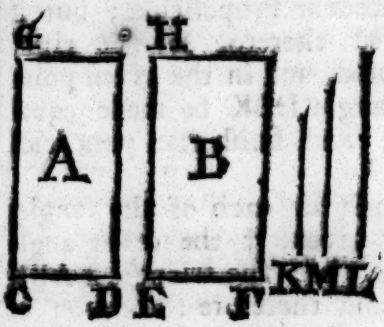
Conſtr. Let the given right line K be expoſed, and ſeeing that the reaſon of A to B is given, let the ſame be made of K to L; therefore the reaſon of K to L is alſo given.

Demonſtr. But K is given; therefore *f* L is alſo given. Again, ſeeing that the reaſon of CD to EF is given, let the ſame be made of K to M: Therefore the reaſon of K to M is given. But K is given, therefore *g* M is alſo given; and therefore the reaſon of L to M is given. Now ſeeing that A is equiangled to B, *h* the reaſon of the ſaid A to B is compounded of that of the ſides, that is to ſay, of CD to EF, and of CG to EH. But alſo the reaſon of K to L is compounded of K to M, and of M to L; therefore the reaſon compounded of CD to EF, and of CG to EH, is the ſame with that which is compounded of K to M, and of M to L (the reaſon of K to L being the ſame as of A to B:) But the reaſon of CD to EF is the ſame as of K to M: Therefore the other reaſon of CG to EH is alſo the ſame as of M to L. But the ſaid reaſon of M to L is given: Therefore alſo the reaſon of CG to EH is given.

f 2. *prop.*

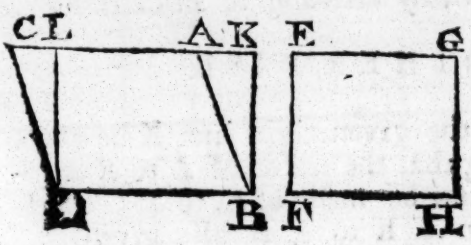
g 1. *prop.*

h 23. 6.



PROP.

PROP. LXIX.



If two parallelograms, CB and EH, having the angles D and F given, and that a side hath also a given reason to a side; in like manner the other side shall have a given reason to the other side.

Constr. Let the reason of BD to FH be also given: I say that the reason of AB to EF is given. For if CB be equiangled to HE, it is manifest by the precedent Proposition; but if it be not equiangled thereto, let the right line DB be constituted, and in the given point B therein, let the angle DBK be made equal to the angle EFH, and finish the parallelogram DK.

Demonstr. Forasmuch as each of the angles BKL and BAK is given, † the other angle
a 40. *prop.* KBA is given: Wherefore the triangle *a* ABK is given by kind; and therefore the reason of AB to BK is given. But the reason of CB to
b 35. *prop.* EH *is* supposed to be given, and *b* CB is equal to DK; therefore the reason of DK to EH is given; and seeing that DK is equiangled to EH, and the reason of the said DK to EH is
c 68. *prop.* given, as also that of DB to FH, *c* the reason of BK to FE is given. But the reason of the
d 29. 1. said BK to BA is also given: Therefore *d* the reason of AB to FE is given.

Scholium.

Scholium.

† *EUCLIDE ſuppoſeth there that a parallelogram having one angle given, all the other angles are alſo given, and as well the antient Interpreters as others, do give the reaſons why, the angle F being given, the other angle E ſhall be alſo given, it being the remainder of two right angles, for that on the parallel lines EG and FH there doth fall the line EF, which makes* e *the two internal angles (of the ſame part) F and G, equal* e 29. 1. *to two right angles. But to thoſe angles* f *the oppoſite angles G and H are equal, and therefore they are alſo given.* f 34. 1.

From whence it follows that the angles BDC and F being given by ſuppoſition, all the other angles of the two parallelograms CB and EH, are alſo given: Therefore the angle DBK having been made equal to the angle F, the angle K ſhall be equal to the angle E, and given as that is: But the angle BAL which is oppoſite to the given angle BDC, is alſo given; and therefore BAK which is the remainder of two right angles, ſhall be alſo given; in ſuch ſort as in the triangle ABK, the two angles BAK and BKA are given, as EUCLIDE doth declare in this place.

PROP. LXX.

If of two parallelograms AB and EH, the ſides about the equal angles, or about the unequal angles (yet nevertheleſs given angles have to one another a given reaſon, to wit (AC to EF, and CB to FH) alſo the ſame parallelograms AB and EH ſhall have to one another a given reaſon.

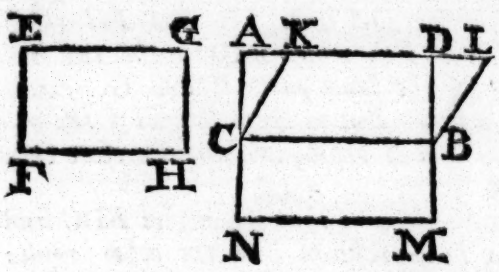
Conſtr. For let AB be prolonged to EH, and on the right line CB let the parallelogram CM be applied equal to the parallelogram EH, in ſuch

ſuch ſort as AC may be direct to CN; that is to ſay, that AC and CN make one right line; and by conſequence DB ſhall be *a* directly with BM.

a *ſch.* 68. *prop.*

Demonſtr. Foraſmuch then as CM is equiangled and equal to EH, the ſides about the equal angles ſhall be reciprocally *b* proportional: Wherefore as BC is to HF, ſo is FE to NC. But the reaſon of BC to HF is given: Therefore the reaſon of FE to NC is alſo given.

b 14. 6.



But the reaſon of AC to the ſame EF is given: Therefore *c* the reaſon of AC to NC is alſo given. Wherefore the reaſon of AB to CM is given; (for it is the ſame *d* as of AC to CN.) But CM is equal to EH: Therefore the reaſon of AB to EH is given.

c 8. *prop.*

d 1. 6.

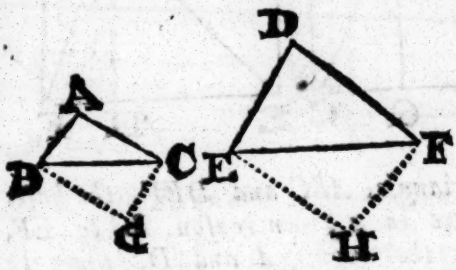
Conſtr. Now ſuppoſe AB not to be equiangled to EH, and on the right line CB, and in the given point C therein: Let there be conſtituted the angle BCK, equal to the given angle F, and ſo finiſh the parallelogram CL.

Demonſtr. Foraſmuch as the angle ACB is given, and the angle BCK alſo given, the remaining angle ACK is given: Therefore the triangle ACK *e* is given by kind; and therefore the reaſon of AC to CK is given: But the reaſon of AC to EF is alſo given: Therefore the reaſon of CK to EF is given. But the reaſon of BC to HF is alſo given, and the angle BCK is equal to the angle F; therefore

e 40. *prop.*

fore (by the firſt part of this Propoſition) the reaſon of CL to EH is given. But to the ſaid CL, AB is equal: Therefore the reaſon of AB to EH is given.

PROP. LXXI.



If of two triangles ABC and DEF, the ſides about the equal angles A and D, or elſe about the unequal angles (yet neverthelеſs given angles) have to one another a given reaſon (to wit AB to DE and AC to DF) the ſame triangles ſhall have alſo to one another a given reaſon ABC to DEF.

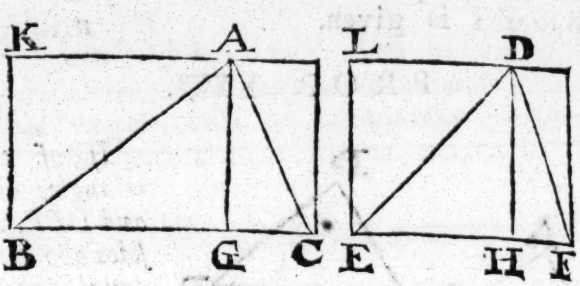
Conſtr. Let the parallelogram AG and DH be finiſhed.

Demonſtr. Seeing that the two parallelograms AG and DH, have the ſides about the equal angles A and D, or elſe about the unequal angles (nevertheleſs given) have a given reaſon to one another, the reaſon *a* of the parallelogram AG to the parallelogram DH is given. But the triangle ABC is the half of the parallelogram AG *b* and the triangle DEF the half of the parallelogram DH. Therefore the reaſon of the triangle ABC to the triangle DEF is given.

a 70. *prop.*

b 34. *prop.*

PROP. LXXII.



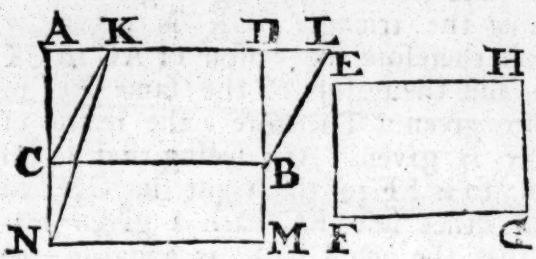
If of two triangles ABC and DEF, the baſes BC and EF, are in a given reaſon, BC to EF, and that from the angles A and D, there be drawn to thoſe baſes the right lines AG and DH, making the equal angles AGC and DHF, or elſe unequal (yet nevertheleſs given) which ſhall have to one another given reaſons AG to DH, thoſe triangles ABC and DEF ſhall have alſo a given reaſon to one another, to wit, ABC to DEF.

Conſtr. For let the parallelogram KC and LF be finiſhed.

Demonſtr. Foraſmuch as the angles AGC and DHF are equal, or unequal (yet given) and
a 29. 1. that the angle AGC *a* is equal to the angle KBC. But the angle DHF equal to the angle LEF, the angles at the points B and E are equal, or elſe unequal (yet given) and for that the reaſon of AG to DH is given, and AG is equal to KB. But DH is equal to LE, alſo the reaſon of KB to LE is given. But the reaſon of BC to EF is alſo given, and the angles at the points B and E are equal, or
b 7c. *prop.* elſe unequal (yet given:) Therefore *b* the reaſon of the parallelogram KC to the parallelogram LF is given; and therefore the reaſon of the triangle ABC to the triangle DEF is gi-
c 41. 1. ven, ſeeing thoſe triangles *c* are the one half of the parallelograms.

PROP.

PROP. LXXIII.



If of two parallelograms A-B and EG, the ſides about the equal angles C and F, or elſe about the unequal angles (but nevertheleſs given) are in ſuch ſort to one another, that as the ſide CB of the firſt, is to the ſide FG of the ſecond; ſo the other ſide EF of the ſecond, is to ſome other right line CN. But that the other ſide AC hath alſo to the ſame right line CN a given reaſon, thoſe parallelograms will have alſo to one another a given reaſon AB to EG.

Conſtr. For in the firſt place, let the parallelogram AB be equiangled to EG, and having placed CN directly to AC: Let the parallelogram CM be finiſhed.

Demonſtr. Foraſmuch then as CB or NM its equal, is to FG, ſo is EF to CN, and that the angles N and F are equal (for N is equal to the angle ACB, which is put equal to F) the parallelograms CM and EG *a* are equal: a 14. 6.
But as AC to CN, ſo *b* the parallelogram AB b 1. 6.
is to the parallelogram CM or EG: Therefore ſeeing that the reaſon of AC to CN is given, the reaſon of AB to EG is alſo given.

Conſtr. 2. Now ſuppoſe the parallelogram AB not to be equiangled to the parallelogram EG, and let there be conſtituted at the given point C in the line CB, the angle BCK, equal to the angle EFG, and ſo finiſh the parallelogram CL.

Demonſtr. 2. Seeing that each of the angles ACB and KCB is given, the remaining angle
c *ſch.* 69. *prop.* ACK is alſo given. But *c* the angle CAK is given, as alſo the remaining angle AKC:
d 40. *prop.* Therefore *d* the triangle ACK is given by kind; and therefore the reaſon of AC to CK is given. But the reaſon of the ſame AC to
e 8. *prop.* CN is alſo given: Therefore *e* the reaſon of CK to CN is given. And ſeeing that as CB is to FG; ſo is EF to the right line CN, to which the other ſide KC hath a given reaſon, and that the angle BCK is equal to the angle F, the reaſon of the parallelogram CL to the parallelogram EG is given (by the firſt part of this Propoſition) but the parallelogram CL is equal to the parallelogram AB: Therefore the reaſon of the parallelogram AB to the parallelogram EG is given.

PROP. LXXIV.

If two parallelograms (as in the former figure) AB and EG, in equal angles C and F, or elſe in unequal angles (yet nevertheleſs given angles) have a given reaſon to one another, as one ſide CB of the firſt ſhall be to one ſide FG of the ſecond, ſo the other ſide EF of the ſecond, ſhall be to that to the which the other ſide AC of the firſt hath a given reaſon.

Conſtr. For either AB is equiangled or not; ſuppoſe it in the firſt place to be equiangled, and to the right line BC let there be applied the parallelogram CM, equal to the parallelogram EG, and ſo poſited, as that
a *ſch.* 68. *prop.* AC and CN may be direct: Therefore *a* DB and BM ſhall be alſo direct (that is as one right line.)

Demonſtr. Seeing that the reaſon of AB to EG is given, and that CM is equal to EG, the reaſon of AB to CM is alſo given; and therefore the reaſon of AC to CN is given (ſeeing

(ſeeing AB is to CM, *b* as AC is to CN;) and for that CM is equal and equiangled to EG, the ſides about the equal angles of the parallelograms CM and EG, *c* are reciprocally proportional; and therefore as CB is to FG, ſo is EF to CN. But the reaſon of AC to CN is given: Therefore as CB is to FG, ſo is EF to that to which AC hath a given reaſon.

b 1. 6.

c 14. 6.

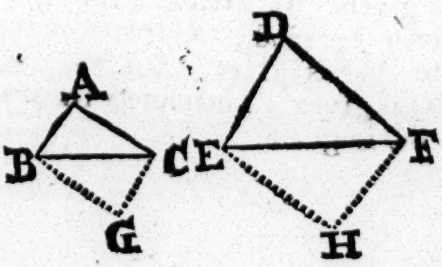
Conſtr. 2. Now ſuppoſe AB not to be equiangled to EG, and in the given point C of the line CB, let there be conſtituted the angle BCK equal to the angle EFG, and finiſh the parallelogram CL.

Demonſtr. 2. Seeing then that the reaſon of AB to EG is given, and *d* that AB is equal to CL, alſo the reaſon of CL to EG is given, and the angle BCK is equal to the angle F, and therefore CL *e* is equiangled to EG: Therefore (by the firſt part of this Propoſition) as CB is to FG, ſo is EF to that to which CK hath a given reaſon. But the reaſon of AC to CK is given; (as appears by what hath been demonſtrated in the latter part of the precedent Propoſition.) Therefore as CB is to FG, ſo is EF to that to which AC hath a given reaſon.

d 36. 1.

e *ſch.* 69. *prop.*

PROP. LXXV.



If two triangles ABC and DEF, in equal angles A and D, or elſe unequal (yet nevertheleſs given) have to one another a given reaſon, as the ſide AB of the firſt, ſhall be to the ſide DE of the ſecond, ſo the other

other ſide DF of the ſecond, ſhall be to that right line to the which the other ſide AC of the firſt hath a given reaſon.

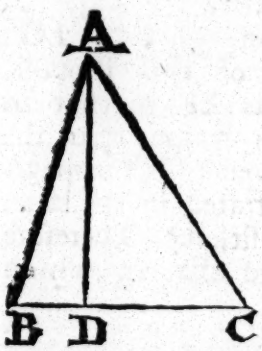
Conſtr. For let the parallelograms AG and DH be finiſhed.

Demonſtr. Foraſmuch as the reaſon of the triangle ABC to the triangle DEF is given, alſo the reaſon of the parallelogram AG to the parallelogram DH is given.

Seeing therefore that the two parallelograms AG and DH in equal angles, or unequal angles (nevertheleſs given) have to one another a given reaſon; as *a* AB is to DE, ſo is DF to that to which AC hath a given reaſon.

a 74 *prop.*

PROP. LXXVI.



If from the top A of a triangle ABC, given by kind, there be drawn to the baſe BC, a perpendicular line AD, that line AD ſhall have to the baſe BC a given reaſon.

Demonſtr. For ſeeing that the triangle ABC is given by kind, the reaſon of AB to BC is given; and the angle B is alſo given. But the angle ADB is given; therefore the other angle BAD is given. Wherefore *a* the triangle ABD is given by kind; and therefore the reaſon of AB to AD is given. But the reaſon of AB to BC is given: Therefore *b* the reaſon of AD to BC is given.

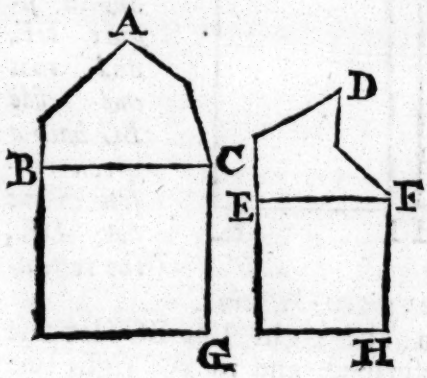
a 40. *prop.*

b 8. *prop.*

PROP.

PROP. LXXVII.

If two figures ABC and DEF, given by kind, have to another a given reaſon, the reaſon alſo ſhall be given of which you pleaſe of the ſides of one of the figures, to which you pleaſe of the ſides of the other figure.



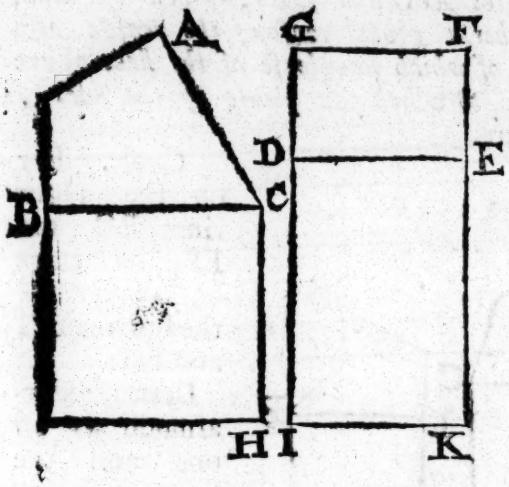
Conſtr. For on the right lines BC and EF, let there be deſcribed the ſquares BG and EH.

Demonſtr. For-aſmuch as on one and the ſame right line BC, are deſcribed two figures ABC and BG, given by kind, *a* the reaſon of the ſaid ABC to BG is given: In like manner, the reaſon of DEF to EH is given; and ſeeing that the reaſon of ABC to DEF is given, and alſo that of the ſame figure ABC to BG; and again the reaſon of DEF to EH: *b* the reaſon of BG to EH is given; and therefore the reaſon of BC to EF is alſo given.

a 49. *prop.*

b 8. *prop.*

PROP. LXXVIII.



If a given figure ABC, hath a given reaſon to ſome rectangled figure DF, and that one ſide BC hath a given reaſon to one ſide DE, the rectangled figure DF is given by kind.

Conſtr. For on the right line BC let the ſquare BH be deſcribed, and to the right line DE, let the parallelogram DK be applied equal to BH, in ſuch manner, as that GD and DI
a *ſch.* 68. *prop.* may be placed directly, *a* and by conſequence FE and EK alſo directly.

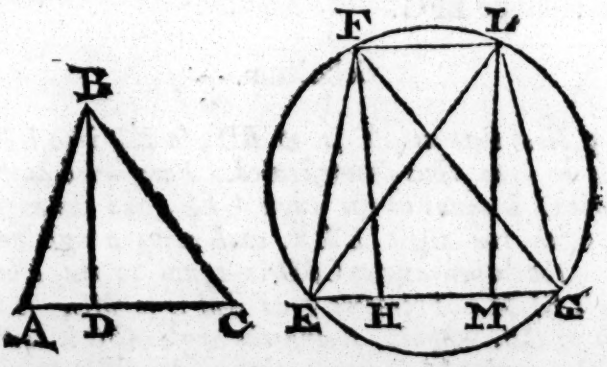
Demonſtr. Therefore ſeeing that on one and the ſame right line BC are deſcribed the two rectiline figures ABC and BH, given by kind,
b 49. *prop.* *b* the reaſon of ABC to BH is given. But the reaſon of the ſaid ABC to DF is alſo
c 8. *prop.* given: Therefore *c* the reaſon of BH to DF is given. But BH is equal to DK: Therefore the reaſon of DK to DF is alſo given. And ſeeing that BH is equal and equiangled to DK, both the one and the other being rect-
d 14. 6. angles, *d* the ſides of thoſe figures are reciprocally proportional; and as BC is to DE, ſo is DI to CH. But by ſuppoſition, the reaſon of BC to DE is given; therefore alſo the reaſon of DI to CH is given; but the reaſon of

of DI to DG is alſo given: (for DI is to DG *e* as DK to DF:) Therefore *f* the reaſon of DG to CH is given. But CH is equal to BC, ſeeing that BH is a ſquare; therefore the reaſon of BC to DG is given. But the reaſon of the ſame BC to DE is alſo given; therefore the reaſon of DE to DG is given, and the angle at D is a right angle: Therefore *g* DF is given by kind.

e 1. 6.
f 8. *prop.*
g *ſch.* 61. *prop.*

PROP. LXXIX.

If two triangles ABC and EFG, have an angle B equal to an angle F. But from the equal angles B and F there be drawn perpendiculars BD and FH, to the baſes AC and EG; and that as the baſe AC of the firſt triangle ABC, is to the perpendicular BD, ſo alſo the baſe EG of the other triangle EFG, is to the perpendicular FH, thoſe triangles ABC and EFG are equiangled.



Conſtr. For about the triangle EFG let there be deſcribed the circle FFLG, then on the right line EG, and in the point E given therein, let there be made the angle GEL, equal to the angle C, and let FL and LG be drawn, and the perpendicular LM.

Demonſtr.

Demonſtr. Seeing then that the angle GEL is equal to the angle C, and the angle ELG
a 21. 3. is equal to the angle EFG, *a* they being in one and the ſame ſegment of the circle; the third angle EGL is equal to the third angle A. Wherefore the triangle ABC is alike to the triangle ELG, and the perpendiculars BD and LM are drawn: Therefore † as AC is to BD, ſo is EG to LM; but by ſuppoſition as AC is to BD; ſo is EG to FH: Therefore
b 7. 5. *b* LM is equal to FH. But the ſaid LM
c 28. 1. is *c* parallel to FH: Therefore *d* FL is alſo pa-
d 33. 1. rallel to EG; and therefore the angle FLE
e 29. 1. *e* is equal to the angle LEG. But the angle C is alſo equal to the ſaid angle LEG, and
f 21. 3. the angle FLE to the angle FGE *f*: Therefore alſo the angle C is equal to the angle FGE. But by ſuppoſition the angle ABC is equal to the angle EFG: Therefore the third angle BAC is equal to the third angle FEG: Wherefore the triangle ABC is equiangled to the triangle EFG.

Scholium.

† *Now that as AC is to BD, ſo EG is to LM, it is by ſome thus demonſtrated. Foraſmuch as the angle C is equal to the angle GEL, and the angle BDC to the angle LME, each being a right angle, the other angle CBD is equal to the other*
g 4. 6. *angle ELM: Therefore* g *as EM is to ML, ſo is CD to DB. Again, ſeeing the angle ABC is equal to the angle ELG, and the angle CBD to the angle ELM, the remaining angle ABD is equal to the remaining angle MLG; but the angle ADB is alſo equal to the angle LMG; and therefore the third angle A is equal to the third angle*
h 4. 6. *LGM: Therefore* h *as AD is to DB, ſo is GM to ML. But it hath been demonſtrated that as*

CD

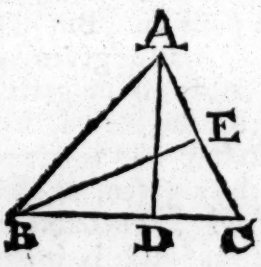
CD is to DB, ſo is EM to ML: Therefore as AC is to BD, ſo is EG to LM. i 14. 5.

PROP. LXXX.

If a triangle ABC hath one angle A given, and that the rectangle contained under the ſides AB and AC, compriſing the given angle A, hath a given reaſon to the ſquare of the other ſide BC, the triangle ABC is given by kind.

Conſtr. For from the points A and B, let there be drawn the perpendiculars AD and BE.

Demonſtr. Foraſmuch as the angle BAE is given, and alſo the angle AEB, the triangle ABE is given by *a* kind; and therefore the reaſon of AB to BE is given: Therefore the reaſon of the rectangle of AB and AC to the rectangle of BE and AC is alſo given (for it is the ſame reaſon *b* as of AB to BE.) But the rectangle of AC and BE is equal to the rectangle of BC and AD; for that each of thoſe rectangles is *c* double to the triangle ABC. Therefore the reaſon of the rectangle of AB and AC to the rectangle of BC and AD is alſo given. But the reaſon of the rectangle of AB and AC to the ſquare of BC is given: Therefore *d* alſo the reaſon of the rectangle of BC and AD to the ſquare of BC is given; and therefore the reaſon of the right line EC to the right line AD is given. (For that *e* the rectangle is to the ſquare as AD to BC.) Now let the right line FG given by poſition and magnitude, be expoſed; and thereon let there be deſcribed the ſegment of a circle FIG; capable



a 40. *prop.*

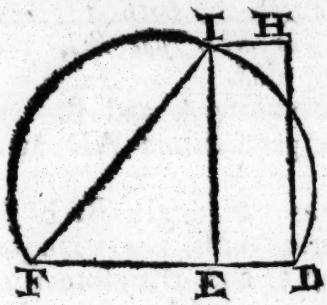
b 1. 6.

c 41. 1.

d 8. *prop.*

e 1. 6.

capable of an angle equal to the angle A. And ſeeing the ſaid angle A is given, alſo the angle in the ſegment FLG ſhall be given; and therefore *f* the ſame ſegment is given by poſition. From the point G let there be erected at right angles on the line FG, the line GH, which *g* is given by poſition: Let it be ſo made, that as BC is to AD, ſo FG may be to GH; and ſeeing that the reaſon of BC to AD is given, alſo that of FG to GH is given. But FG is given: Therefore *h* GH is given by magnitude. But it is alſo given by poſition, and the point G is given: Therefore the point H is *i* alſo given. Now by the point H let there be drawn HI, parallel to FG, and that line HI ſhall be given by *k* poſition. But the ſegment of the circle FIG is alſo given by poſition. Therefore *l* the point I is given. Let the right lines IF and IG be drawn, and the perpendicular IK: Therefore IK is given by poſition. But the point I is given, as alſo each of the points F and G: Therefore *m* each of the lines FG, FI, and IG is given by poſition and magnitude: Wherefore *n* the triangle FIG is given by kind; and ſeeing that as BC is to AE, ſo is FG to GH, and *o* that to GH, IK is equal, as BC is to AE, ſo is FG to IK, and the angle A is equal to the angle FIG: Therefore *p* the triangle ABC is equiangled to the triangle FIG. But FIG is given by kind: Therefore alſo the triangle ABC is given by kind.



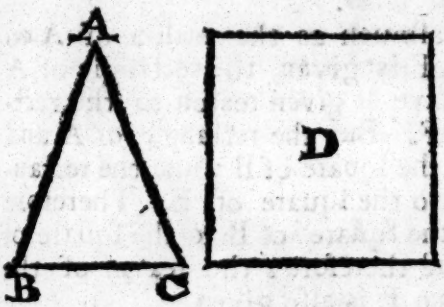
f 8. *def.*
g 4. *def.*
h 2. *prop.*
i 27. *prop.*
k 28. *prop.*
l 25. *prop.*
m 26. *prop.*
n 39. *prop.*
o 34. *prop.*
p 79. *prop.*

OTHER-

OTHERWISE.

Conſtr. Let the triangle ABC, whoſe angle A is given, and the reaſon of the rectangle contained under AB and AC, to the ſquare of BC be given: I ſay that the triangle ABC is given by kind.

Demonſtr. For ſeeing the angle A is given, that ſpace of which the ſquare of the line compounded of BAC is greater than the ſquare of BC, *q* hath a given reaſon to the triangle ABC. Now let that ſpace be D: Therefore the reaſon of D to the triangle ABC is given. But the reaſon of the triangle ABC to the rectangle of AB and AC is given; *r* ſeeing the angle A is given: Therefore *s* the reaſon of the ſpace D to the rectangle of AB and AC is given. But the reaſon of the rectangle of AB and AC to the ſquare of BC is alſo given: Therefore *s* the reaſon of the ſpace D to the ſquare of BC is given. Wherefore by compounding, *t* the reaſon of the ſpace D, with the ſquare of BC to the ſaid ſquare of BC is given: Therefore the reaſon of the ſquare of the line compounded of BAC, to the ſquare of BC is given; (for that the ſpace D with the ſquare of BC is equal to the ſquare of the line compounded of BAC;) and therefore *u* the reaſon of the ſaid line compounded of BAC to BC is given. But the angle A is alſo given: There-



q 67. *prop.*

r 66. *prop.*

s 8. *prop.*

t 6. *prop.*

u *ſch.* 52. *prop.*

x 46. *prop.* Therefore *x* the triangle ABC is given by kind.

PROP. LXXXI.

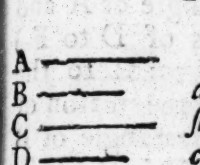
If of three right lines A, B, and C, proportional to three other proportional right lines D, E, and F, the extremes A and D, C and F, are in a given reaſon (to wit, as A to D, and C to F,) alſo the means B and E ſhall be in a given reaſon, and if one extreme hath a given reaſon to an extreme, and the mean to the mean, the other will have alſo a given reaſon to the other.

Demonſtr. Foraſmuch as the reaſon of A to D, and of C to F is given, the rectangle of A
a 70. *prop.* and D *a* ſhall have a given reaſon to the rectangle of C and F. But the rectangle of A and
b 17. 6. D is equal *b* to the ſquare of B; and the rectangle of C and F to the ſquare of E. Therefore the reaſon of the ſquare of B to the ſquare of
c *ſch.* 52. *prop.* E is given; and therefore *c* the reaſon of the line B to the line E is alſo given.

Again, Let the reaſon of A to D, and B to E, be given: I ſay that the reaſon of C to F is alſo given. For ſeeing that the reaſon of A to D, and of B to E is given, alſo the reaſon
d 50. *prop.* of the ſquare of B *d* to the ſquare of E is given. But the ſquare of B is equal to the rectangle of A and C, and the ſquare of E to the rectangle of D and F: Therefore the reaſon of the rectangle of A and C to the rectangle of D and F is given. But the reaſon of a ſide A to a ſide D is given: Therefore *e* the reaſon of
e 68. *prop.* the other ſide C to the other ſide F is alſo given.

PROP.

PROP. LXXXII.

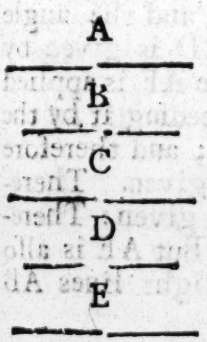


If there be four right lines A, B, C, and D, proportional, as the first A, ſhall be to that line to which the ſecond B hath a given reaſon, ſo the third C, ſhall be to that to which the fourth D hath a given reaſon.

Conſtr. Let E be the line to which B hath a given reaſon, and let it be ſo as that B may be to E, as D is to F.

Demonſtr. Now the reaſon of B to E is given, therefore alſo the reaſon of D to F is given. And ſeeing that as A is to B, ſo is C to D. And again, as B is to E, ſo is D to F, by reaſon of equality as A is to E, ſo is C to F. But E is that line to which B hath a given reaſon, and F that to which D alſo hath a given reaſon: Therefore as A is to that to which B hath a given reaſon, ſo C is to that to which D hath a given reaſon.

PROP. LXXXIII.

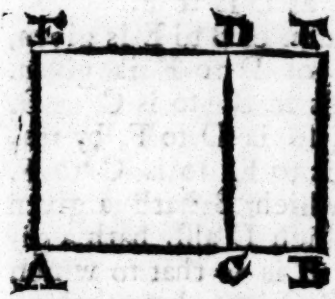


If four right lines A, B, C, and D, are in ſuch ſort to one another, that of any three of them A, B, and C, and a fourth E, taken proportional, to which that line D, which remains of the four lines, hath a given reaſon, the four lines A, B, C, and E, are proportional; as the fourth D is to the third C, ſo the ſecond B ſhall be to that to which the firſt A hath a given reaſon.

Demonſtr. Foraſmuch as A is to B as C is to E, the rectangle contained under A and E *a* is equal to the rectangle contained under B and C; a 16. 6.

C; and ſeeing that the reaſon of D to E is given, alſo ſhall be given the reaſon of the rectangle of A and D to the rectangle of A and
b 1. 6. E (for *b* it is the ſame reaſon as of D to E.) But the rectangle of A and E is equal to the rectangle of B and C: Therefore the reaſon of the rectangle of A and D to the rectangle of B
c 74. *prop.* and C is given. Wherefore *c* as D is to C, ſo is B to that to which A hath a given reaſon.

PROP. LXXXIV.

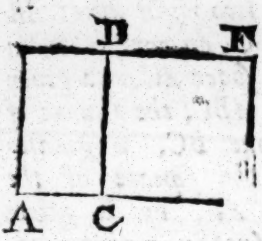


If two right lines AB and AE comprehending a given ſpace AF in a given angle BAE, and that the one AB be greater than the other AE by a given line CB, alſo each of the lines AB and AE is given.

Demonſtr. For ſeeing that AB is greater than AE by the given line CB, the remainder AC is equal to AE: Finiſh the parallelogram AD. Therefore ſeeing that AE is equal to AC, the reaſon of AE to AC is given, and the angle
a *ſch.* 61. *prop.* A is alſo given: Therefore *a* AD is given by kind. Wherefore the given ſpace AF is applied to the given right line CB, exceeding it by the given figure AD given by kind; and therefore
b 59. *prop.* *b* the breadth of the exceſs is given. Therefore AC is given. But CB is alſo given: Therefore the whole AB is given. But AE is alſo given: Therefore each of the right lines AB and AE is given.

PROP.

PROP. LXXXV.



If two right lines AC and CD, do comprehend a given ſpace AD in a given angle ACD, the line compounded of thoſe lines AC and CD is given, alſo each of thoſe lines AC and CD is given.

Conſtr. For let AC be prolonged to the point B, and let CB be put equal to CD, then by the point B let BF be drawn parallel to CD, and ſo finiſh the parallelogram CF.

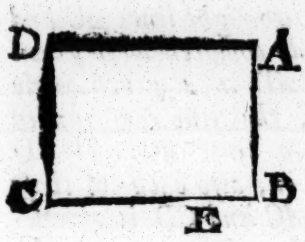
Demonſtr. Seeing then that CB is equal to CD, and the angle DCB is given; for that angle that follows is the given angle; and therefore *a* the parallelogram DB is given by kind; and again, ſeeing that the line compounded of ACD is given, and CB is equal to CD, alſo AB is given. And thus to the right line AB there is applied the given ſpace AD, deficient by the figure DB given by kind; and therefore *b* the breadths of the defects are alſo given: Therefore the right lines DC and CB are given. But the compounded line ACD is alſo given: Therefore *c* each of the lines AC and CD is given.

a *ſch.* 61. *prop.*

b 58. *prop.*

c 4. *prop.*

PROP. LXXXVI.



If two right lines AB and BC, do comprehend a given ſpace AC, in a given angle ABC, the ſquare of the one BC, is greater than the ſquare of the other AB, by a given ſpace (yet in a given reaſon) alſo each of thoſe lines AB and BC ſhall be given.

Demonſtr. For ſeeing that the ſquare of BC is greater than the ſquare of AB by a given ſpace (yet in a certain reaſon.) Let the given ſpace be taken away, that is to ſay, the rectangle contained under CB and BE: Therefore
a 11. *def.* *a* the reaſon of the remainder, *b* which is the
b 2. 2. rectangle contained under BC and CE to the ſquare of AB is given. And foraſmuch as the rectangle † under AB and BC is given, and
c 1. *prop.* alſo that of CB and BE, their *c* reaſon is given. But as the rectangle under AB and BC is to
d 1. 6. the rectangle under CB and EB, *d* ſo AB is to BE; and therefore the reaſon of AB to BE
e 50. *prop.* is given: Wherefore *e* the reaſon of the ſquare of AB to the ſquare of BE is alſo given. But the reaſon of the ſquare of AB to the rectan-
f 8. *prop.* gle under BC and CE is given: Therefore *f* alſo the reaſon of the rectangle under BC and CE to the ſquare of BE is given. Wherefore the reaſon of four times the rectangle under BC and CE to the ſquare of BE is given;
g 8. *prop.* and by compounding, *g* the reaſon of four times the rectangle under BC and CE, with the ſquare of BE to the ſquare of BE is given. But four times the rectangle of BC and CE,
h 8. 2. with the ſquare of BE, *h* is the ſquare of the compound line BCE: Therefore the reaſon of the

ſquare of the compound line BCE to the ſquare of BE is given: Wherefore *i* the reaſon of the line compounded of BC and CE to BE is given, and by compounding, *k* the reaſon of the compound of the lines BC, CE, and BE, that is to ſay, the double of BC to BE is given; and therefore the reaſon of the only line BC to BE is alſo given. But as BC is to BE, *l* ſo the rectangle under BC and BE is to the ſquare of BE: Therefore the reaſon of the rectangle under BC and BE to the ſquare of BE is given. But the rectangle of BC and BE is given: Therefore *m* the ſquare of BE is alſo given, and conſequently the line BE is given. Wherefore BC is alſo given, ſeeing that the reaſon of BE to BC is given. But the ſpace AC is given, and alſo the angle B: Therefore *n* AB is given. Wherefore each of the lines AB and BC is given.

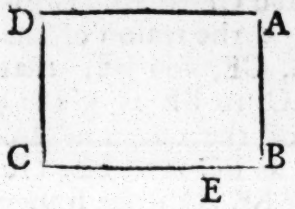
i 54. *prop.*
k 6. *prop.*
l 1. 6.
m 2. *prop.*
n 57. *prop.*

Scholium.

† *Inſtead of ſaying in this place* [what is under, *&c.*] *we have uſed this Word Rectangle, it being manifeſt by what follows that ſuch was the intention of* EUCLIDE, *ſeeing he makes uſe in the ſaid Demonſtration of the ſecond and eighth Propoſition of the twelfth Element; and alſo that the ſpace or Parallelogram given being not rectangled, it may be reduced thereto, making on BC, and in the given point B, a right angle CBA, ſo as that there will be two Parallelograms conſtituted on one and the ſame baſe BC, and between the ſame parallels, as in the 69th Propoſition by means whereof this Concluſion is drawn.*

Note, This ſerves alſo for the next Prop.

PROP. LXXXVII.



If two right lines AB and BC, do comprehend a given ſpace AC, in a given angle B, the ſquare of the one BC, is greater than the ſquare of the other AB, by a given ſpace, alſo each of thoſe lines AB and BC ſhall be given.

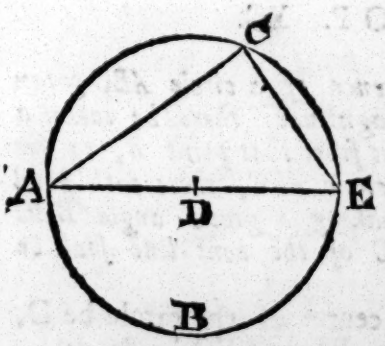
Demonſtr. For ſeeing that the ſquare of BC is greater than the ſquare of AB by a given ſpace: Let the given ſpace be taken away, and let the rectangle be contained under BC and
a 2. 2. BE: Therefore the remainder, *a* which is the rectangle of BC and CE, is equal to the ſquare of AB. And ſeeing that the rectangle of BC and BE is given, and alſo the ſpace or rectangle AC; the reaſon of the ſaid rectangle of
b 1. 6. BC and BE to AC is given. But as *b* the rectangle of BC and BE is to the rectangle of AB and BC, ſo is BE to AB: Therefore the reaſon of BE to AB is given, and therefore
c 50. *prop.* *c* the reaſon of the ſquare of the ſaid DE to the ſquare of AB is alſo given. But to that ſquare of AB the rectangle of BC and CE is equal: Therefore the reaſon of the ſaid rectangle of BC and CE to the ſquare of BE is given; and therefore the reaſon of the quadruple of the ſaid rectangle of BC and CE to the ſquare of BE is alſo given; and by compound-
d 6. *prop.* ing, *d* the reaſon of four times the rectangle of BC and CE, with the ſquare of BE, to the ſaid ſquare of BE is given. But four times the rectangle of BC and CE, with the ſquare
e 8. 2. of BE, *e* is the ſquare of the compound line BCE: Therefore the reaſon of the ſquare of that compound line BCE to the ſquare of BE

is

is alſo given; and therefore the reaſon *f* of the compound line BCE to BE is given. Wherefore by compounding, *g* the reaſon of the ſaid compound line BCE and EB, that is to ſay, twice BC to BE is alſo given; therefore the reaſon of the only line BC to BE is given. But the reaſon of the ſame BE to AB is alſo given: Therefore *h* the reaſon of AB to BC is given. And ſeeing that the reaſon of BC to BE is given, and that as the ſaid BC is to BE, ſo the ſquare of BC *i* to the rectangle of BC and BE, the reaſon of the ſquare of BC to the rectangle of BC and BE is alſo given. But the ſaid rectangle of BC and BE is given, it being that which was taken away, and which was given. Therefore the ſquare of BC *k* is given, and therefore the line BC is given. But the reaſon of the ſame BC to BA is given, therefore AB is alſo given.

f 54. *prop.*
g 6. *prop.*
h 8. *prop.*
i 1. 6.
k 2. *prop.*

PROP. LXXXVIII.



If in a circle ABC, given by magnitude, there be drawn a right line AC, which ſhall take away a ſegment ABC, which doth comprehend a given angle AEC, that line AC is given by magnitude.

Conſtr. For let D be the center of the circle; and let the diameter thereof ADE be drawn, and let EC be joined.

Demonſtr. Foraſmuch as the angle ACE is given, for *a* it is a right angle. But the angle AEC is alſo given, and therefore the other angle CAE is given. Wherefore the triangle

a 31. 3.

b 40. *prop.* ACE *b* is given by kind; and therefore the reaſon of EA to AC is given. But AE is given by magnitude, ſeeing that the circle ABC
c 2. *prop.* is given by magnitude. Therefore *c* AC is alſo given by magnitude.

PROP. LXXXIX.

If in a circle ABC, given by magnitude, there be drawn a right line AC, given by magnitude, that line AC will take away a ſegment ABC, comprehending a given angle.

Conſtr. For having taken the point D for the center of the circle, let the diameter ADE be drawn, as alſo the right line EC.

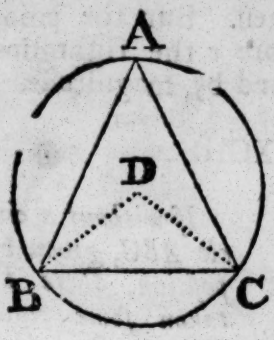
Demonſtr. Foraſmuch as each of the right lines AE and AC are given, the reaſon of the
a 1. *prop.* line AE to AC *a* is given; and the angle ACE
b 43. *prop.* is a right angle: Therefore *b* the triangle ACE is given by kind, and therefore the angle AEC is given.

PROP. XC.

If in the circumference of a circle ABC given by poſition, and by magnitude, there be taken a given point B, and that from that point B, to the circumference of the circle ABC, there doth bend a right line BAC, making a given angle BAC the other extremity C of the bent line ſhall be given.

Conſtr. For let the center of the circle be D, and let the right lines BD and BC be drawn.

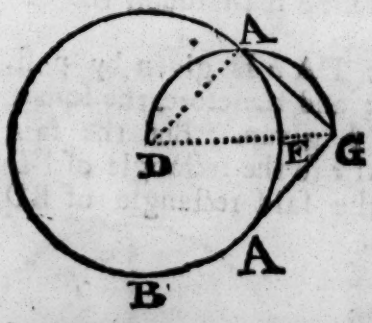
Demonſtr.



Demonſtr. Foraſmuch as each point B and D is given, the right line BD, *a* is given by poſition; and ſeeing that the angle BAC is given, the angle BDC is alſo given. Wherefore to the right line BD given by poſition, and in the point D given therein, there is drawn the right line C D; which makes the given angle BDC; and therefore *b* the line DC is given by poſition. But the circle ABC is given by poſition and magnitude: Therefore *c* the right line DC is given by poſition and by magnitude. But the point D is given: Therefore *d* the point C is alſo given.

a 26. *prop.*
b 29. *prop.*
c 6. *def.*
d 27. *prop.*

PROP. XCI.



If from a given point C, there be drawn a right line CA, which ſhall touch a circle AB, given by poſition; that line CA is given by poſition and by magnitude.

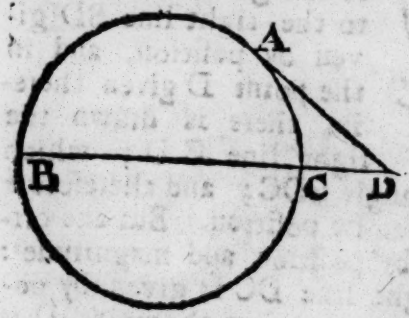
Conſtr. For having taken the point D for the center of the circle, let the right lines DA and DC be drawn.

Demonſtr. Foraſmuch as each point C and D is given, the right line CD *a* is given by poſition and by magnitude. But the angle CAD *b* is a right angle; and therefore the ſemicircle deſcribed on CD ſhall paſs by the point A: Let it then paſs by that point, and let the ſemicircle be DAC: Foraſmuch as the ſame

a 26. *prop.*
b 18. 3.

c 6. *def.* DAC *c* is given by poſition, and alſo the circle
d 25. *prop.* ABE, *d* the point A is given. But the point
e 26. *prop.* C is alſo given: Therefore *e* the right line AC is given by poſition and by magnitude.

PROP. XCII.



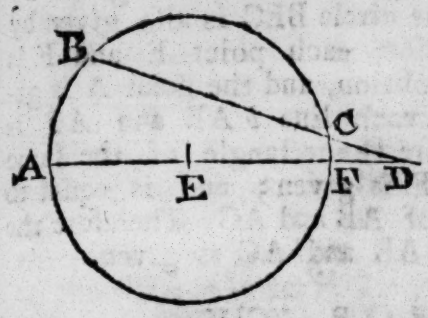
If without a circle ABC, given by poſition, there be taken ſome point D, and from that given point there be drawn a right line DB, cutting the circle, the rectangle compriſed under the whole line BD, and the part DC, between the point D, and the circumference convex AC ſhall be given.

Conſtr. For from the point D let the right line DA be drawn, which ſhall touch the circle in the point A.

a 91. *prop.* *Demonſtr.* Therefore DA *a* is given by poſition and magnitude; and therefore the ſquare
b 52. *prop.* of the ſaid DA is *b* given. But the ſaid
c 36. 3. ſquare of DA is equal *c* to the rectangle of BD and DC: Therefore the ſaid rectangle of BD and DC is alſo given.

OTHER-

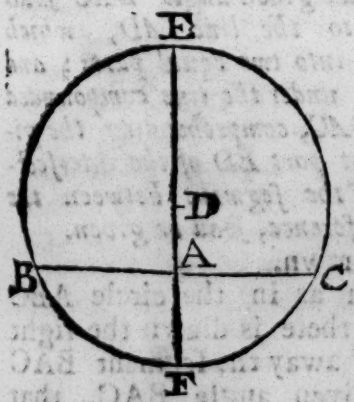
OTHERWISE.



Constr. Let E be the center of the circle, and by the ſame center let there be drawn from the point D the right line DA.

Demonſtr. For-aſmuch as each point D and E is given, the right line DE is *d* given by po- d 26. *prop.* ſition and by magnitude. But the circle ABC is given by poſition and by magnitude: Therefore each point A and F *e* is given, and the point D is alſo given; and therefore *d* each e 25. *prop.* line AD and FD is given. Wherefore the rectangle of the lines AD and DF is alſo given. But the ſaid rectangle of AD and DF is equal to the rectangle of DB and DC: Therefore the rectangle of DB and DC is given.

PROP. XCIII.



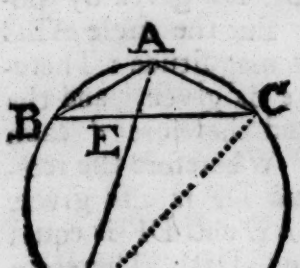
If in a circle given by poſition there be taken a given point A, and by that point A there be drawn a right line BC to the circle, the rectangle compriſed under the ſegments of the ſame line BC ſhall be given.

Constr. For let D be taken for the center of the circle, and having

having drawn the right line AD prolong it to the points E and F.

Demonſtr. Foraſmuch as each point A and
a 26. *prop.* D is given, the right line AD *a* is given by poſition. But the circle BEC is alſo given by poſition: Therefore each point E and F is alſo given by poſition, and the point A is gi-
b 35. 3. ven. Wherefore each line *b* AE and AF is given: Therefore the rectangle of the ſame lines AE and AF is given; and is equal to the rectangle *b* of AB and AC: Therefore the ſaid rectangle of AB and AC is given.

PROP. XCIV.



If in a circle ABC, given by magnitude, there be drawn a right line BC, which doth take away a ſegment which doth comprehend a given angle ABC, and that the ſaid angle being in the ſegment is cut into two equal parts, the line compounded of the right lines BA and AC, which comprehend the given angle BAC ſhall have a given reaſon to the line AD, which doth divide that angle into two equal parts; and the rectangle contained under the line compounded of thoſe lines BA and AC, comprehending the given angle BAC, and that part ED of the interſecting line which is below the ſegment between the baſe BC and the circumference, ſhall be given.

Conſtr. Let BD be drawn.

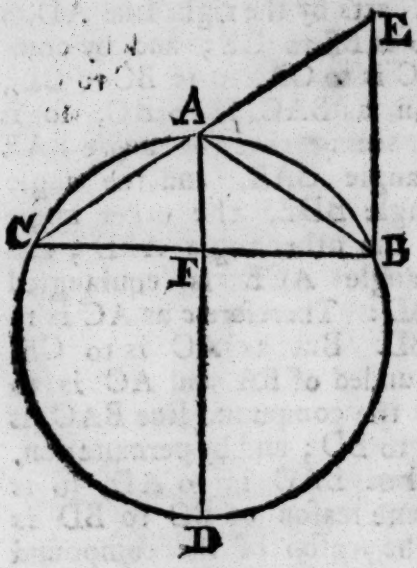
Demonſtr. Foraſmuch as in the circle ABC given by magnitude, there is drawn the right line BC, which takes away the ſegment BAC comprehending the given angle BAC, that line

line BC *a* is given; and therefore BD is alſo given: Therefore the reaſon of BC to BD *b* is given. And ſeeing that the given angle BAC is cut in two equal parts by the right line AD, as *c* BA is to CA, ſo is BE to CE; and by compounding, as BAC is to CA, ſo is BC to CE; and by permutation, as BAC is to BC, ſo is CA to CE. And ſeeing that the angle BAE is equal to the angle CAE, and the angle ACE *d* to the angle BDE, the other angle AEC is equal to the other angle ABD; and therefore the triangle ACE is equiangled to the triangle ABD: Therefore *e* as AC is to CE, ſo is AD to BD. But as AC is to CE, ſo the line compounded of BA and AC is to BC: Therefore as the compound line BAC is to BC, ſo is AD to BD; and by permutation, as the compound line BAC is to AD, ſo is BC to BD. But the reaſon of BC to BD is given: Therefore the reaſon of the compound line BAC to AD is alſo given. Moreover, I ſay that the rectangle under the compound line BAC and ED is given. For ſeeing that the triangle AEC is equiangled to the triangle BDE, (for the angle ACE *d* is equal to the angle BDE, and the angle AEC *f* to the angle BED) as BD is to DE, ſo is AC to CE. But as AC is to CE, ſo is alſo the compound line BAC to BC: Therefore as the compound line BAC is to BC, ſo is BD to DE. Wherefore the rectangle of the compound line BAC and DE *g* is equal to the rectangle of BC and BD. But the rectangle of BC and BD is given, (for that thoſe lines BC and BD are given:) Therefore the rectangle under the compound line BAC and ED is alſo given.

a 88. *prop.*
b 1. *prop.*
c 3. 6.
d 21. 3.
e 4. 6.
f 15. 1.
g 16. 6.

OTHER-

OTHERWISE.



Conſtr. Let CA be prolonged to the point E, and let AE be put equal to BA, and let BE and BD be joined.

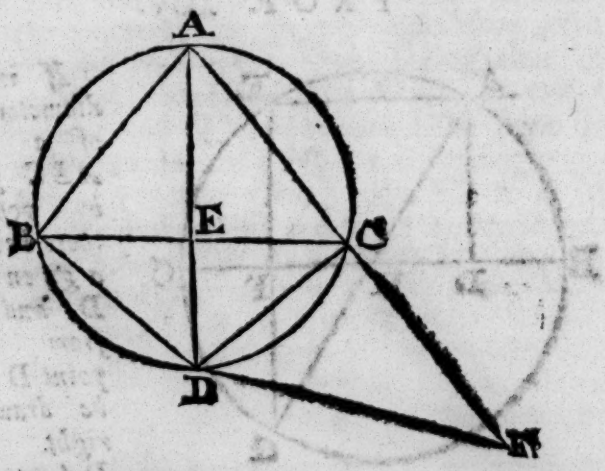
Demonſtr. Forasmuch as the angle BAC is double to each of the angles CAD and AEB (for the angle BAC is cut into two equal parts by the line AD, and equal *h* to the two angles ABE and AEB, which *i* are equal) the angle ABE is equal to the angle CAD, that is to ſay, *k* to the angle CBD; adding therefore the common angle ABC, the whole angle ABD ſhall be equal to the whole angle FBE. But the angle ACB is *k* equal to the angle ADB: Therefore the third angle AEB is equal to the third angle BAD; and therefore the triangle CEB is equiangled to the triangle ABD: Wherefore as CE is to CB, ſo is AD to BD. But the right line CE is compounded of the two lines CA and AB: Therefo e as the compound line BAC is to CB, ſo is AD to BD; and by permutation, as the compound line BAC is to AD, ſo is CB to BD. But the reaſon of CB to BD is given, ſeeing that each of thoſe lines is given: Therefore the reaſon of the compound line BAC to AD is alſo given. And ſeeing that the

h 32. 1.
i 5. 1.
k 21. 3.

the triangle CEB is equiangled to the triangle FBD (for the angle AFC is equal *l* to the angle BFD, and the angle ECB *m* to the angle ADB) as EC is to CB, so is BD to DF. But EC is equal to the compound line BAC: Therefore as the compound line BAC is to CB, so is BD to DF. Wherefore *n* the rectangle of the compound line BAC and DF is equal to the rectangle of CB and BD. But the rectangle of CB and BD is given, considering that each of the lines CB and BD is given: Therefore the rectangle of the compound line BAC and DF is given.

l 21. 3.
m 16. 6.
n 16. 6.

OTHERWISE.



Constr. Let AC be prolonged to F, and let CF be put equal to AB, and let the right lines BD and DF be drawn.

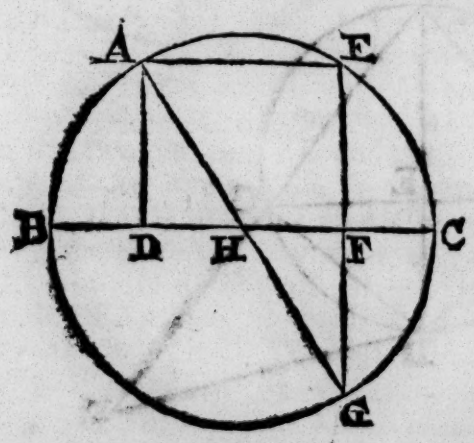
Demonstr. Forasmuch as BA is equal to CF, and *o* BD to DC, the two sides AB and BD are equal to the two sides CD and DF, each to his corresponding side, and the angle ABD is equal to the angle DCF, *p* seeing that the four

o 26, 29, 3.
p 22. 3.

four sided figure ABDC is within the circle: Therefore the base AD is *q* equal to the base DF, and the angle DAB to the angle DFC. But the angle BAD is given, (being the half of the given angle BAC) Therefore the angle DFC is so also. But DAF is also given: Therefore the triangle ADF is given by kind. Wherefore the reason of FA to AD is given. But AF is the compound of BA and AC, for that CF is equal to AB: Therefore the reason of the compound line BAC to AD is given: The same Demonstration will serve to shew that the rectangle contained under the compound line BAC and ED is given also.

q 4. 1.

PROP. XCV.



If in the diameter BC of a circle ABC given by position, there be taken a given point D, and that from that point D there be drawn a right line DA, to the circumference of the circle. But from the section of the said line there be drawn a right line AE, perpendicular thereto, and by the point E where that perpendicular doth meet with the circumference, there be drawn a parallel EF, to the first line drawn AD, that point F in which the parallel meets with the diameter, is given; and the rectangle contained under the parallel lines AD and EF is also given.

Constr.

Conſtr. Let the right line EF be prolonged to the point G, and let the right line AG be drawn.

Demonſtr. Foraſmuch as the angle AEG is a right angle, the right line AG is the diameter of the circle. But BC is alſo the diameter: Therefore the point H is the center of the circle. Now the point D is given; and therefore *a* the line DH is given by magnitude. But ſeeing that AD is parallel to EG, and AH equal to GH; *b* DH is equal to FH, and AD to FG; (for the angles AHD and FHG *c* are equal, and DAH and FGH *d* are alſo equal.) But the line DH is given: Therefore FH is alſo given. But each of thoſe lines DH and HF is alſo given by poſition, and the point H is given: Therefore *e* the point F is alſo given. And ſeeing that in the circle ABC given by poſition, is taken the given point F, and through the ſame is drawn the right line EFG; the rectangle under EF and FG *f* is given. But FG is equal to AD. Therefore the rectangle comprehended under AD and EF is given. Which was to be demonſtrated.

a 26. *prop.*
b 26. *prop.*
c 15. 1.
d 29. 1.
e 27. *prop.*
f 93. *prop.*

The End of EUCLIDE's *DATA.*

A BRIEF TREATISE, Added by *FLUSSAS*, OF Regular Solids.

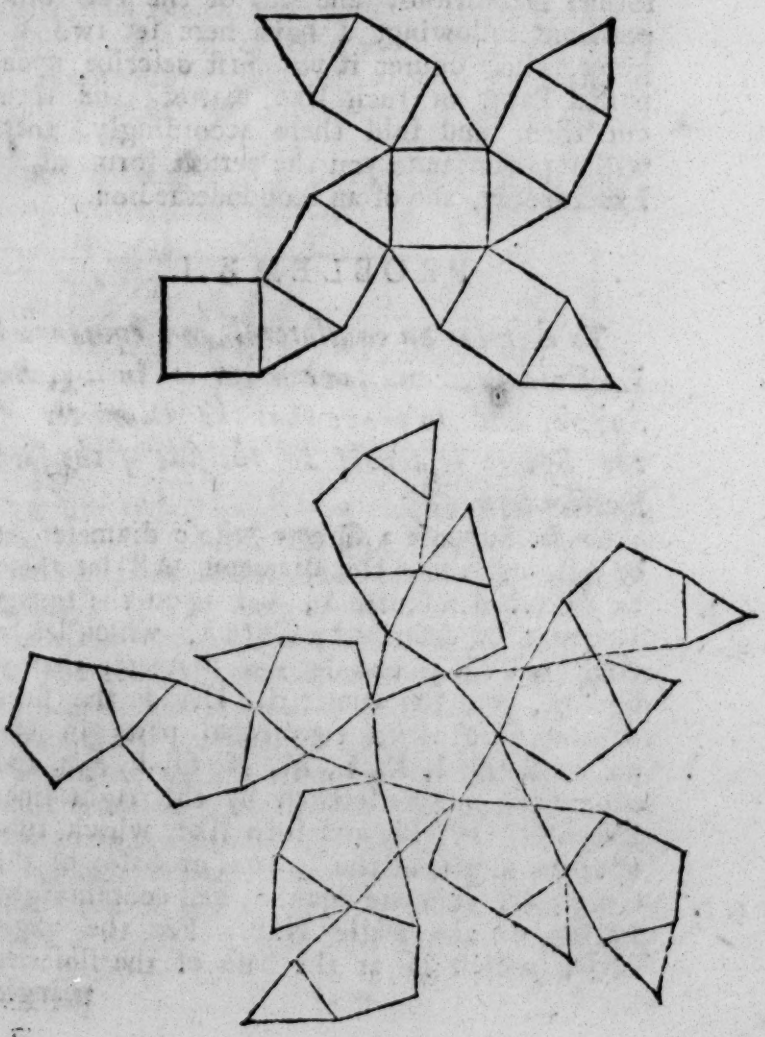
REgular Solids are ſaid to be compoſed and mix'd when each of them is transformed into other Solids, keeping ſtill the form, number and inclination of the baſes, which they before had to one another; ſome of which yet are transformed into mix'd Solids, and other ſome into ſimple.

Into mixt, as a Dodecaedron and an Icoſaedron, which are transformed or altered, if you divide their ſides into two equal parts, and take away the ſolid angles ſubtended of plain ſuperficial figures, made by the lines coupling thoſe middle ſections; for the Solid remaining after the taking away of thoſe ſolid angles, is called an Icoſidodecaedron. If you divide the ſides of a Cube and of an Octoedron into two equal parts, and couple the ſections, the ſolid angles ſubtended of the plain ſuperficies made by the coupling lines, being taken away, there ſhall be left a ſolid, which is called an Exoctoedron. So that both of a Dodecaedron and alſo of an Icoſaedron, the Solid which is made ſhall be called an Icoſidodecaedron; and likewiſe the Solid made of a Cube, and alſo of an Octoedron, ſhall be called an Exoctoedron. But the other Solid, to wit, a Pyramis or Tetraedron, is transformed into a ſimple Solid; for if you divide into two equal parts each of the ſides of the pyramis, triangles deſcribed of the lines which couple the ſections, and ſubtending and taking away the ſolid angles of the Pyramis, are equal and like unto the equilateral triangles left in each of the baſes, of all which triangles is produced an Octoedron, to wit, a ſimple, and not a compoſed Solid. For the Octoedron hath four baſes, like in number, form, and mutual inclination with the baſes of the pyramis, and hath the other four baſes with like ſituation oppoſite and parallel to the former. Wherefore the application of the pyramis taken twice, maketh a ſimple Octoedron, as the other Solids make a mix'd compound Solid.

DEFI-

DEFINITIONS.

I. *An Exoctoedron is a ſolid figure contained of ſix equal ſquares, and eight equilateral and equal triangles.*



II. *An Icosidodecaedron is a solid figure contained under twelve equilateral, equal, and equiangled Pentagons, and twenty equal and equilateral triangles.*

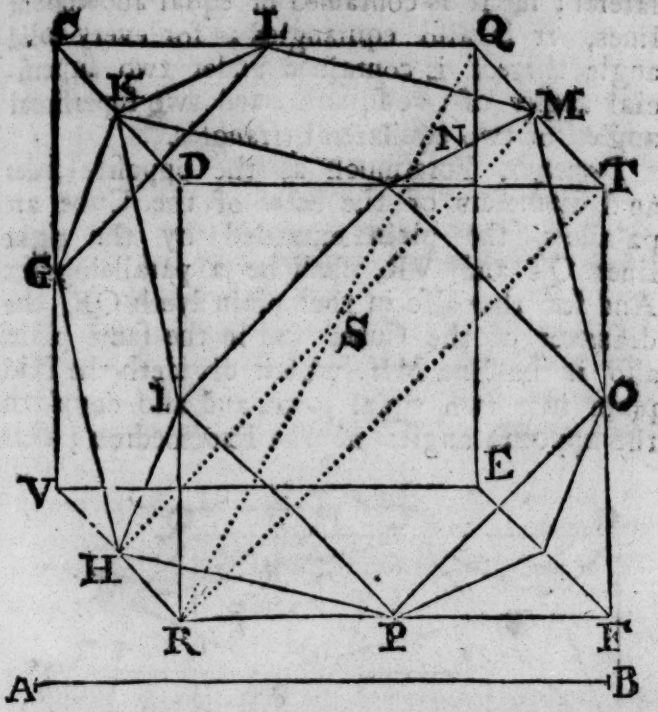
For the better Understanding of the two former Definitions, and also of the two Propositions following, I have here set two figures, whose figures if you first describe upon pasted Paper or such like matter, and then cut them and fold them accordingly, they will represent unto you the perfect forms of an Exoctoedron, and of an Icosidodecaedron.

PROBLEME I.

To describe an equilateral and equiangled Exoctoedron, and to contain it in a given Sphere, and to prove that the Diameter of the Sphere is double to the side of the said Exoctoedron.

Constr. Suppose a Sphere whose diameter let be AB, and about the diameter AB let there
a 6. 4. be described a square *a*, and upon the square
b 15. 13. let there be described a Cube *b*, which let be CDEFQTVR; and let the diameter thereof be QR, and the center S. Divide the sides of the Cube into two equal parts in the points G, H, I, K, L, M, N, O, P, *&c.* and couple the middle sections by the right lines IN, NO, OP, PI, and such like, which subtend the angles of the squares or bases of the
c 4. 1. Cube; and they are equal *c*, and contain right angles, as the angle NIP. For the angle NID, which is at the base of the Isosceles triangle

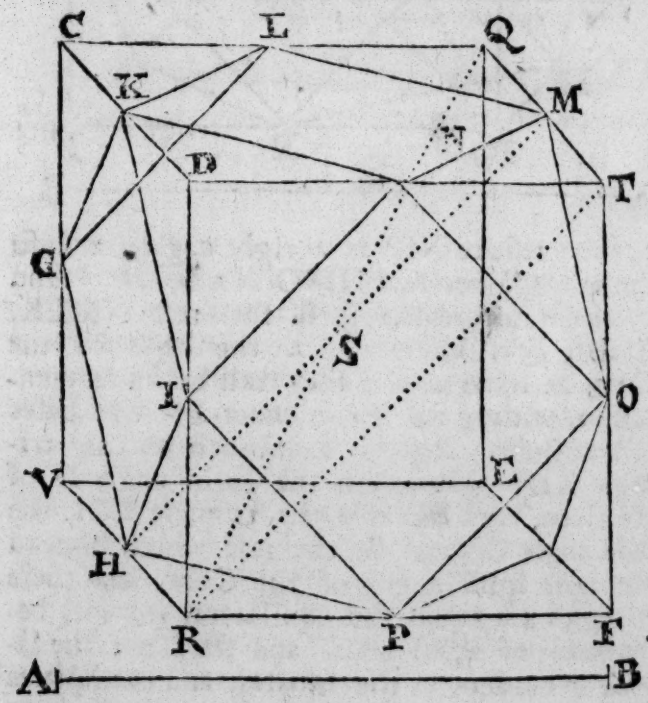
triangle NDI, is the half of a right angle, and ſo likewiſe is the oppoſite angle RIP. Where-



fore the reſidue NIP is a right angle, and ſo the reſt. Wherefore NIPO is a ſquare. And by the ſame reaſon ſhall the reſt NMLK, KGHI, *&c.* inſcribed in the baſes of the Cube, be ſquares, and they ſhall be ſix in number, according to the number of the baſes of the Cube. Again, foraſmuch as the triangle KIN ſubtendeth the ſolid angle D of the Cube, and likewiſe the triangle KGL the ſolid angle C, and ſo the reſt which ſubtend the right ſolid angles of the Cube, and theſe triangles are equal and equilateral (to wit) being made of equal ſides, and they are the limits or borders of the ſquares, and the ſquares the limits or borders of them; as hath been

before proved. Wherefore LMNOPHGK is an Exoctoedron by the definition, and is equilateral; for it is contained of equal ſubtendant lines, it is alſo equiangled; for every ſolid angle thereof is contained under two ſuperficial angles of two ſquares, and two ſuperficial angles of two equilateral triangles.

Demonſtr. Foraſmuch as the oppoſite ſides and diameters of the baſes of the Cube are parallels, the plain extended by the right lines QT and VR, ſhall be a parallelogram. And for that alſo in that plain lyeth QR, the diameter of the Cube, and in the ſame plain alſo is the line MH, which divideth the ſaid plain into two equal parts, and alſo coupleth the oppoſite angles of the Exoctoedron; this



line MH therefore divideth the diameter into two

two equal parts *d*; and alſo divideth it ſelf in the ſame point, which let be S, into two equal parts *e*. And by the ſame reaſon may we prove that the reſt of the lines which couple the oppoſite angles of the Exoctoedron, do in S the center of the Cube, divide one another into two equal parts, for each of the angles of the Exoctoedron are ſet in each of the baſes of the Cube. Wherefore making the center the point S, with the diſtance SH or SM deſcribe a Sphere, and it ſhall touch every one of the angles equidiſtant from the point S.

d *cor.* 34. 1.

e 4. 1.

And foraſmuch as AB the diameter of the ſphere given, is put equal to the diameter of the baſe of the cube, to wit, to the line RT, and the ſame line RT is equal to the line MH *f*, which line MH coupling the oppoſite angles of the Exoctoedron, is drawn by the center. Wherefore it is the diameter of the Sphere given which containeth the Exoctoedron.

f 33. 1.

Laſtly, foraſmuch as in the triangle RFT, the line PO doth cut the ſides into two equal parts, it ſhall cut them proportionally with the baſes, to wit, as FR is to FP, ſo ſhall RT be to PO *g*. But FR is double to FP by ſuppoſition: Wherefore RT, or the diameter HM, is alſo double to the line PO, the ſide of the Exoctoedron. Wherefore we have deſcribed, *&c.* Which was required to be done.

g 2. 6.

PROBLEME II.

To deſcribe an equilateral and equiangled Icoſidodecaedron, and to comprehend it in a ſphere given, and to prove that the

diameter being divided by an extreme and mean proportion, maketh the greater ſegment double to the ſide of the Icoſi-dodecaedron.

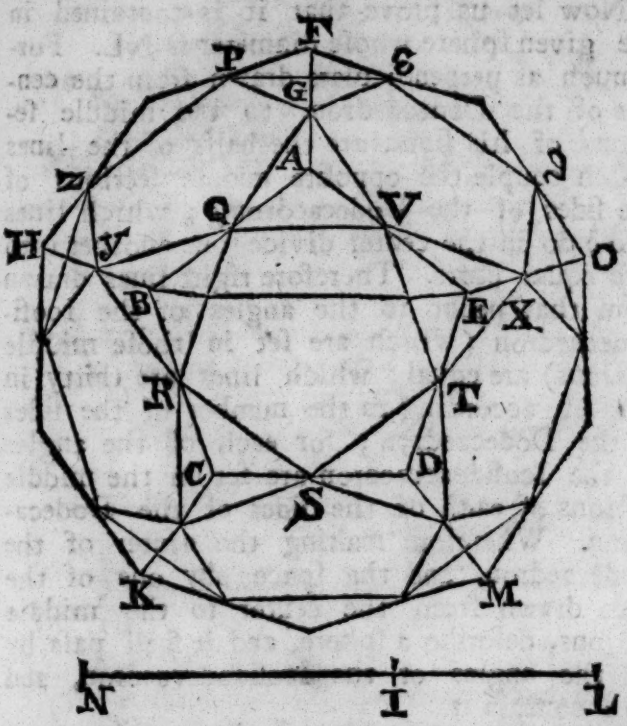
Conſtr. Suppoſe that the diameter of the
a 30. 6. ſphere given be NL, *a* divide the line NL, by an extreme and mean proportion in the point I, and the greater ſegment thereof let be NI; and upon the line NI deſcribe a
b 15. 13. Cube *b*; and about this Cube let there be cir-
c 17. 13. cumſcribed a Dodecaedron *c*; and let the ſame be ABCDEFHKMO, and divide each of the ſides into two equal parts in the points Q, R, S, T, V, X, Y, Z, P, ε, ϑ, G, *&c.* and couple the ſections with right lines, which ſhall ſubtend the angles of the Pentagons, as the lines PG, GV, VQ, QY, YR, RQ, VT, TX, XV, and ſo the reſt.

Demonſtr. Foraſmuch as theſe lines ſubtend equal angles of the Pentagons, and thoſe equal angles are contained of equal ſides, to wit, of the halfs of the ſides of the Pentagons;
d 4. 1. therefore thoſe ſubtending lines are equal *d*. Wherefore the triangles GQV, YQR, and VXT, and the reſt, which take away ſolid angles of the Dodecaedron, are equilateral.

Again, foraſmuch as in every Pentagon are deſcribed five equal right lines, coupling the middle ſections of the ſides, as are the lines QV, VT, TS, SR, and RQ, they deſcribe a Pentagon in the plain of the Pentagon of the Dodecaedron. And the ſaid Pentagon is contained in a circle, to wit, whoſe center is the center of a Pentagon of the Dodecaedron. For the lines drawn from that center to the angles of this Pentagon are equal, for that they are perpendiculars upon the baſes

cut.

cut e. Wherefore the Pentagon QRSTV, is e 12. 4.
equiangled f. And by the ſame reaſon may f 11. 4.



the reſt of the Pentagons deſcribed in the baſes of the Dodecaedron, be proved equal and like.

Wherefore thoſe Pentagons are twelve in number: And foraſmuch as the equal and like triangles do ſubtend and take away twenty ſolid angles of the Dodecaedron; therefore the ſaid triangles ſhall be twenty in number. Wherefore we have deſcribed an Icoſidodecaedron by the Definition, which Icoſidodecaedron is equilateral; for that all the ſides of the triangles are equal and common with the Pentagons; and it is alſo equiangled. For each of the ſolid angles is made of two ſuperficial

perficial angles of an equilateral Pentagon, and of two ſuperficial angles of an equilateral triangle.

Now let us prove that it is contained in the given ſphere whoſe diameter is NL. Foraſmuch as perpendiculars drawn from the centers of the Dodecaedron, to the middle ſections of his ſides, are the halfs of the lines which couple the oppoſite middle ſections of the ſides of the Dodecaedron *g*; which lines alſo *h* do in the center divide one another into two equal parts. Therefore right lines drawn from that point to the angles of the Icoſidodecaedron (which are ſet in thoſe middle ſections) are equal; which lines are thirty in number, according to the number of the ſides of the Dodecaedron; for each of the angles of the Icoſidodecaedron are ſet in the middle ſections of each of the ſides of the Dodecaedron. Wherefore making the center of the Dodecaedron, and the ſpace any one of the lines drawn from the center to the middle ſections, deſcribe a ſphere, and it ſhall paſs by all the angles of the Icoſidodecaedron, and ſhall contain it.

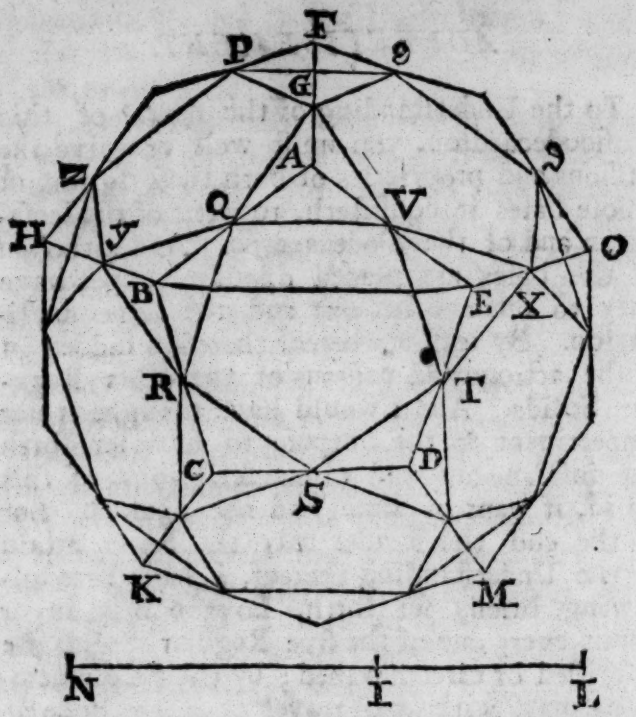
g 3 *cor.* 17. 13.

h *idem.*

And foraſmuch as the diameter of this ſolid, is that right line whoſe greater ſegment is the ſide of the Cube inſcribed in the Dodecaedron *i*, which ſide is NI by ſuppoſition. Wherefore that ſolid is contained in the ſphere given, whoſe diameter is put to be the line NL.

i 4. *cor.* 17. 13.

Now



Now let us prove that the greater ſegment of the diameter is double to QV the ſide of the ſolid. Foraſmuch as the ſides of the triangle AEB, are in the points Q and V divided into two equal parts, the lines QV and BE are parallels *k*. Wherefore as AE is to AV, ſo is EB to VQ *l*. But the line AE is double to the line AV. Wherefore the line BE is double to the line QV *m*. Now the line BE is equal to NI, or to the ſide of the Cube *n*; which line NI is the greater ſegment of the diameter NL. Wherefore the greater ſegment of the diameter given is double to the ſide of the Icoſidodecaedron inſcribed in the given ſphere. Wherefore, We have deſcribed, *&c.* Which was required to be done.

k *cor.* 39. 1.
l 2. 6.
m 4. 6.
n 2 *cor.* of 17. 13.

ADVER-

ADVERTISEMENT.

To the Underſtanding of the nature of this Icoſidodecaedron, you muſt well conceive the paſſions and proprieties of both theſe ſolids, of whoſe baſes it conſiſteth, to wit, of the Icoſaedron and of the Dodecaedron. And altho' in it the baſes are placed oppoſitely, yet have they to one another one and the ſame inclination. By reaſon whereof there lye hidden in it the actions and paſſions of the other Regular Solids. And I would have thought it not impertinent to the purpoſe to have ſet forth the inſcriptions and circumſcriptions of this Solid, if want of time had not hindred. But to the end the Reader may the better attain to the Underſtanding thereof, I have here following briefly ſet forth, how it may in or about every one of the five Regular Solids be inſcribed or circumſcribed; by the help whereof he may, with ſmall travel or rather none at all, having well poiſed and conſidered the Demonſtrations appertaining to the foreſaid five Regular Solids, demonſtrate both the inſcription of the ſaid Solids in it, and the inſcription of it in the ſaid Solids.

Of the Inſcriptions and Circumſcriptions of an Icoſidodecaedron.

An Icoſidodecaedron may contain the other five regular bodies. For it will receive the angles of a Dodecaedron in the centers of the triangles which ſubtend the ſolid angles of the Dodecaedron, which ſolid angles are twenty in number, and are placed in the ſame order in which the ſolid angles of the Dodecaedron taken away, or ſubtended by them, are. And by that reaſon it ſhall receive a Cube and a Pyramis

Pyramis contained in the Dodecaedron, when as the angles of the one, are ſet in the angles of the other.

An Icoſidodecaedron receiveth an Octoedron, in the angles cutting the ſix oppoſite ſections of the Dodecaedron, even as if it were a ſimple Dodecaedron.

And it containeth an Icoſaedron, placing the twelve angles of the Icoſaedron in the ſame centers of the twelve Pentagons.

It may alſo by the ſame reaſon be inſcribed in each of the five regular bodies, to wit, in a Pyramis, if you place four triangular baſes concentrical with four baſes of the Pyramis, after the ſame manner that you inſcribed an Icoſaedron in a Pyramis; ſo likewiſe may it be inſcribed in an Octoedron, if you make eight baſes thereof concentrical with the eight baſes of the Octoedron. It ſhall alſo be inſcribed in a Cube, if you place the angles which receive the Octoedron in it, in the centers of the baſes of the Cube. Again, you ſhall inſcribe it in an Icoſaedron, when the triangles compaſſed in of the Pentagon baſes, are concentrical with the triangles which make a ſolid angle of the Icoſaedron.

Laſtly, It ſhall be inſcribed in a Dodecaedron, if you place each of the angles thereof in the middle ſections of the ſides of the Dodecaedron, according to the order of the Conſtruction thereof.

The oppoſite plain ſuperficies alſo of this ſolid are parallels. For the oppoſite ſolid angles are ſubtended of parallel plain ſuperficies, as well in the angles of the Dodecaedron ſubtended by triangles, as in the angles of the Icoſaedron ſubtended of Pentagons, which thing may eaſily be demonſtrated. Moreover, in this ſolid are infinite properties and paſſions, ſpringing of the ſolids whereof it is compoſed.

Where-

Wherefore it is manifeſt, that a Dodecaedron and an Icoſaedron mixed, are transformed into one and the ſelf ſame ſolid of an Icoſidodecaedron. A Cube alſo and an Octoedron are mixed and altered into another ſolid, to wit, into one and the ſame Exoctoedron. But a Pyramis is transformed into a ſimple and perfect ſolid, to wit, into an Octoedron.

If we will frame theſe two ſolids joined together into one ſolid, this only muſt we obſerve.

In the Pentagon of a Dodecaedron inſcribe a like Pentagon, and let its angles be ſet in the middle ſections of the Pentagon circumſcribed, and then upon the ſaid Pentagon inſcribed, let there be ſet a ſolid angle of an Icoſaedron, and ſo obſerve the ſame order in each of the baſes of the Dodecaedron, and the ſolid angles of the Icoſaedron ſet upon theſe Pentagons ſhall poduce a ſolid conſiſting of the whole Dodecaedron, and whole Icoſaedron. In like ſort, if in every baſe of the Icoſaedron, the ſides being divided into two equal parts, be inſcribed an equilateral triangle, and upon each of thoſe equilateral triangles be ſet a ſolid angle of a Dodecaedron, there ſhall be produced the ſame ſolid conſiſting of the whole Icoſaedron, and of the whole Dodecaedron.

And after the ſame order, if in the baſes of a Cube be inſcribed ſquares ſubtending the ſolid angles of an Octoedron, or in the baſes of an Octoedron be inſcribed equilateral triangles ſubtending the ſolid angles of a Cube, there ſhall be produced a ſolid conſiſting of either of the whole ſolids, to wit, of the whole cube, and of the whole Octoedron.

But equilateral triangles inſcribed in the baſes of a Pyramis, having their angles ſet in the middle ſections of the ſides of the Pyramis,

and

and the ſolid angles of a Pyramis, ſet upon the ſaid equilateral triangles, there ſhall be produced a ſolid conſiſting of two equal and like pyramids.

And now if in theſe ſolids thus compoſed, you take away the ſolid angles, there ſhall be reſtored again the firſt compoſed ſolids, to wit, the ſolid angles taken away from a Dodecaedron and an Icoſaedron compoſed into one, there ſhall be left an Icoſidodecaedron, the ſolid angles taken away from a Cube and an Octoedron compoſed into one ſolid, there ſhall be left an Exoctoedron. Moreover, the ſolid angles taken away from two pyramids compoſed into one ſolid, there ſhall be left an Octoedron.

Of the nature of a trilateral and equilateral Pyramis.

1. A trilateral equilateral Pyramis is divided into two equal parts, by three equal ſquares, which in the center of the Pyramis cut one another into two equal parts, and perpendicularly, and whoſe angles are ſet in the middle ſections of the ſides of the Pyramis.

2. From a Pyramis are taken away four Pyramids like unto the whole, which utterly take away the ſides of the Pyramis, and that which is left is an Octoedron inſcribed in the Pyramis, in which all the ſolids inſcribed in the Pyramis are contained.

3. A perpendicular drawn from the angle of the Pyramis to the baſe, is double to the diameter of the Cube inſcribed in it.

4. And a right line coupling the middle ſections of the oppoſite ſides of the Pyramis is triple to the ſide of the ſame Cube.

5. The

5. The ſide alſo of a Pyramis is triple to the diameter of the baſe of the Cube.

6. Wherefore the ſame ſide of the Pyramis is in power double to the right line which coupleth the middle ſections of the oppoſite ſides.

7. And it is in power ſeſquialter to the perpendicular which is drawn from the angle to the baſe.

8. Wherefore the perpendicular is in power ſeſquitertia to the line which coupleth the middle ſections of the oppoſite ſides.

9. A Pyramis and an Octoedron inſcribed in it, alſo an Icoſaedron inſcribed in the ſame Octoedron, do contain one and the ſame ſphere.

Of the nature of an Octoedron.

1. Four perpendiculars of an Octoedron, drawn in four baſes thereof from two oppoſite angles of the ſaid Octoedron, and coupled together by thoſe four baſes, deſcribe a Rhombus, or Diamond figure; one of whoſe diameters is in power double to the other diameter.

2. For it hath the ſame proportion that the diameter of the Octoedron hath to the ſide of the Octoedron.

3. An Octoedron and an Icoſaedron inſcribed in it, do contain one and the ſame ſphere.

4. The diameter of the ſolid of the Octoedron is in power ſeſquialter to the diameter of the circle which containeth the baſe, and is in power duple ſuperbipartiens tertias (that is, as 8 to 3,) to the perpendicular or ſide of the foreſaid Rhombus; and moreover is

is in length triple to the line which coupleth the centers of the next baſes.

5. The angle of the inclination of the baſes of the Octoedron, doth with the angle of the inclination of the baſes of the Pyramis, make angles equal to two right angles.

Of the nature of a Cube.

1. The diameter of a Cube is in power ſeſquialter to the diameter of his baſe.

2. And is in power triple to his ſide.

3. And unto the line which coupleth the centers of the next baſes, it is in power ſextuple.

4. Again, the ſide of the Cube, is to the ſide of the Icoſaedron inſcribed in it, as the whole is to the greater ſegment.

5. Unto the ſide of the Dodecaedron, it is as the whole is to the leſſer ſegment.

6. Unto the ſide of the Octoedron it is in power duple.

7. Unto the ſide of the Pyramis it is in power ſubduple.

8. Again, the Cube is triple to the Pyramis, but to the cube the Dodecaedron is in a manner double. Wherefore the ſame Dodecaedron is in a manner ſextuple to the ſaid Pyramis.

Of the nature of the Icoſaedron.

1. Five triangles of an Icoſaedron, do make a ſolid angle, the baſes of which triangles make a Pentagon. If therefore from the oppoſite baſes of the Icoſaedron be taken the other Pentagon by them deſcribed, theſe Pentagons ſhall in ſuch ſort cut the diameter of the Icoſaedron which coupleth the foreſaid oppoſite angles,

that that part which is contained between the planes of these two Pentagons shall be the greater segment, and the residue which is drawn from the plain to the angle, shall be the lesser segment.

2. If the opposite angles of two bases joined together, be coupled by a right line, the greater segment of that right line is the side of the Icosaedron.

3. A line drawn from the center of the Icosaedron to the angles, is in power quintuple to half that line, which is taken between the Pentagons, or of the half of that line, which is drawn from the center of the circle which containeth the foresaid Pentagon, which two lines are therefore equal.

4. The side of the Icosaedron containeth in power either of them, and also the lesser segment, to wit, the line which falleth from the solid angle to the Pentagon.

5. The diameter of the Icosaedron containeth in power the whole line, which coupleth the opposite angles of the bases joined together, and the greater segment thereof, to wit, the side of the Icosaedron.

6. The diameter also is in power quintuple to the line which was taken between the Pentagons, or to the line which is drawn from the center to the circumference of the circle which containeth the Pentagon composed of the sides of the Icosaedron.

7. The dimetient containeth in power the right line which coupleth the centers of the opposite bases of the Icosaedron, and the diameter of the circle which containeth the base.

8. Again, the said dimetient containeth in power the diameter of the circle which containeth the Pentagon, and also the line which is drawn from the center of the same circle to the circumference; that is, it is quintuple

to

to the line drawn from the center to the circumference.

9. The line which coupleth the centers of the oppoſite baſes, containeth in power the line which coupleth the centers of the next baſes, and alſo the reſt of that line of which the ſide of the Cube inſcribed in the Icoſaedron is the greater ſegment.

10. The line which coupleth the middle ſections of the oppoſite ſides, is triple to the ſide of the Dodecaedron inſcribed in it.

11. Wherefore if the ſide of the Icoſaedron, and the greater ſegment thereof be made one line, the third part of the whole is the ſide of the Dodecaedron inſcribed in the Icoſaedron.

Of the Dodecaedron.

1. The diameter of a Dodecaedron containeth in power the ſide of the Dodecaedron, and alſo that right line to which the ſide of the Dodecaedron is the leſſer ſegment, and the ſide of the Cube inſcribed in it is the greater ſegment, which line is that which ſubtendeth the angle of the inclination of the baſes, contained under two perpendiculars of the baſes of the Dodecaedron.

2. If there be taken two baſes of the Dodecaedron, diſtant from one another by the length of one of the ſides, a right line coupling their centers being divided by an extreme and mean proportion, maketh the greater ſegment the right line which coupleth the centers of the next baſes.

3. If by the centers of five baſes ſet upon one baſe, be drawn a plain ſuperficies, and by the centers of the baſes which are ſet upon the oppoſite baſe, be drawn alſo a plain ſuperficies, and then be drawn a right line, coupling the centers of the oppoſite baſes,

that right line is ſo cut, that each of his parts ſet without the plain ſuperficies, is the greater ſegment of that part which is contained between the plains.

4. The ſide of the Dodecaedron is the greater ſegment of the line which ſubtendeth the angle of the Pentagon.

5. A perpendicular line drawn from the center of the Dodecaedron to one of the baſes, is in power quintuple to half the line which is between the plains.

6. And therefore the whole line which coupleth the centers of the oppoſite baſes, is in power quintuple to the whole line which is between the ſaid plains.

7. The line which ſubtendeth the angle of the baſe of the Dodecaedron, together with the ſide of the baſe are in power quintuple to the line which is drawn from the center of the circle which containeth the baſe, to the circumference.

8. A ſection of a ſphere containing three baſes of the Dodecaedron, taketh a third part of the diameter of the ſaid ſphere.

9. The ſide of the Dodecaedron and the line which ſubtendeth the angle of the Pentagon, are equal to the right line which coupleth the middle ſections of the oppoſite ſides of the Dodecaedron.

THE

THE THEOREMS OF ARCHIMEDES.

Concerning the Sphere and Cylinder, Inveſtigated by the *Method of Indiviſibles*, and briefly Demonſtrated by the Reverend and Learned Dr. *Iſaac Barrow*.

THE main Deſign of *Archimedes* in his *Treatiſe of the Sphere and Cylinder*, is to reſolve theſe four Problems.

1. *To find the proportion of the ſuperficies of a ſphere to any determinate circle; or to find a circle equal to the ſuperficies of a given ſphere.*

2. *To find the proportion of the ſuperficies of any ſegment of a ſphere to any determined circle; or to find a circle equal to the ſuperficies of any aſſign'd ſegment.*

3. *To find the proportion of the ſphere it ſelf (or of its ſolid content) to any determinate Cone or Cylinder; or to find a Cone or Cylinder equal to a given ſphere.*

4. *To find the proportion of a ſegment of a ſphere to any determinate Cone or Cylinder; or to find a Cone or Cylinder equal to a given ſegment.*

Theſe four Problems *Archimedes* proſecutes ſeparately, and lays down Theorems immediately ſubſervient to their ſolution; but we reduce them to two: For ſince an Hemiſphere is the ſegment of a ſphere, and the method of finding out its relations, in reſpect to the ſuperficies and ſolid content, is comprehended in the general method of inveſtigating the proportion of the ſegments: And from the ſuperficies and ſolid content of an Hemiſphere already found, the double of them, (that is, the ſuperficies and content of the whole ſphere) is at the ſame time given. And indeed 'tis ſuperfluous and foreign from the Laws of good Method, to inveſtigate their relations diſtinctly and ſeparately; ſo that if it were not a crime, I might on this account, blame even *Archimedes* himſelf.

The whole matter therefore is reduc'd to theſe two Problems.

1. *To find the proportion of the ſuperficies of any ſegment of a ſphere, to a determinate circle; or to find a circle equal to the ſuperficies of a given ſegment.*

2. *To find the proportion of the ſolidity of any ſegment of a ſphere to any determinate Cone or Cylinder; or to find a Cone or Cylinder equal to an aſſign'd ſegment of a ſphere.*

I ſhall reſolve theſe Problems by another much eaſier and ſhorter method: In which the order being inverted, firſt, I ſhall ſeek the ſolidity of a ſegment, and from thence deduce its

its ſuperficies; a thing which is in my judgment well worth obſerving, and perform'd, as I know of, by none.

Firſt therefore, for finding the ſolidity of a ſegment, I ſhall lay down two, commonly known and receiv'd Suppoſitions, *viz.*

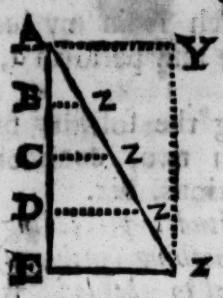
1. *That a ſeries of magnitudes proceeding in Arithmetical Progreſſion from nothing (incluſive) or whoſe common difference is equal to the leaſt magnitude, is ſubduple of as many quantities equal to the greateſt*; (i. e. *ſubduple of the product of the greateſt term and number of terms:*) So that if the *ſum of the terms* be called z, the *greateſt term* g, and the *number of terms* n, then will $z = \frac{ng}{2}$, or $2z = ng$.

The truth of this Propoſition will eaſily appear by expreſſing the ſeries twice, and inverting the order;

0, a, 2a, 3a, 4a.
4a, 3a, 2a, a, 0.

For ſo the difference always being equal to the leaſt quantity, 'twill be evident that each two correſpondent terms taken together are equal to the greateſt term; and alſo, that the ſeries taken twice is equal to the greateſt term repeated as many times as there are terms, *i. e.* the laſt term drawn into the number of terms.

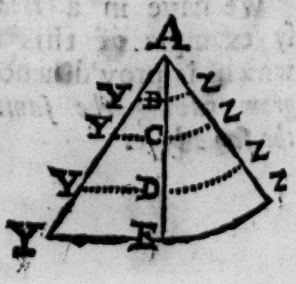
We have in a *triangle* a very clear and eaſy example of this moſt uſeful Propoſition, which is prov'd hence, *to be half a parallelogram having the ſame altitude, and ſtanding on the ſame baſe.*



Suppoſe the altitude AE of the triangle AEZ to be divided into parts indefinitely many and ſmall AB, BC, CD, DE, and parallels BZ, CZ, DZ, EZ, drawn thro' the points of Diviſions; all theſe proceed from *o* in an Arithmetical Progreſſion, and conſequently the ſum of 'em all, that is, the triangle AEZ, is ſubduple of the greateſt EZ drawn into the altitude AE, by which the ſum of the terms is expreſs'd, that is, ſubduple of the Parallelogram EY, whoſe baſe is EZ, and altitude AE.

But the illuſtration of the Rule will conduce more to our deſign by inferring hence, *That a circle is equal to half of the radius drawn into the circumference*, after this manner. Conceive a circle to conſiſt of as many concentric Peripheries as there are points or equal parts indefinitely many and ſmall in the radius. Theſe Peripheries, as well as their radii proceed from the center or nothing in an Arithmetical progreſſion; and therefore their ſum, that is, the whole circle is equal to half the greateſt (or extreme circumference) drawn into the number of terms, that is, the radius.

After the ſame manner we may ſuppoſe the ſector AEZ to conſiſt of as many concentric Arcs BZ, CZ, DZ, EZ, as there are points (or equal parts indefinitely ſmall) in the radius AE, which Arcs, as their radii, proceeding from a point or nothing in an Arithmetical progreſſion, the ſector



alſo

alſo will be equal to half the radius drawn into the extreme Arc EZ. Which may be made evident alſo after this manner: Let us ſuppoſe the right line EY to be perpendicular to the radius AE, draw the right line AY, and from the points B, C, D, of diviſion in the radius, draw BY, CY, DY, parallel to EY, and terminated at AY. Becauſe EY: DY (:: *rad.* AE: *rad.* AD) :: *Arc* EZ: *Arc* DZ. and EY = EZ, then will DY = Arc DZ; and in like manner will CY=CZ, and BY = BZ. whence the triangle AEY will be = to the ſector AEZ, that is, $\frac{AE \times EY}{2}$ ($\frac{AE \times EZ}{2}$) = ſector AEZ. By this means we collect that celebrated Theorem of *Archimedes*, *That a circle is equal to a triangle, whoſe baſe is equal to the radius, and altitude equal to the periphery of the circle*; and that without any inſcription or circumſcription, of figures, by only ſuppoſing that the Area or Superficies of the circle conſiſts of infinitely many concentric Peripheries. Which *method of indiviſibles*, (now firſt of all known to me) ſeems no leſs evident (nay more evident) and perhaps leſs fallacious than that wherein planes are ſuppoſed to conſiſt of parallel right lines, and ſolids of parallel planes; as hereafter ſhall be evident, when we ſhall collect, by this method, the proportions of ſpheric and cylindric ſuperficies to one another, by knowing the ſolid content; and on the other hand, the ſolid content, by knowing the ſuperficies, with admirable facility, and moſt full ſatisfaction in thoſe things which are rigidly gather'd by pure *Geometry*.

2. *Let us ſuppoſe a ſeries of quantities to proceed from* 0 (*incluſive*) *in a duplicate Arithmetic progreſſion, that is*, 0, 1, 4, 9, 16, &c. *the ſquares of*

of numbers in a ſimple Arithmetic progreſſion, 0, 1, 2, 3, 4, &c. *And the triple of this ſeries will always exceed the greateſt term multiplied by the number of terms; but the number of terms increaſing the proportion, continually approximates till at laſt it comes to an equality, when the number of terms is increaſed* in infinitum.

$$3 \times 0 + 1 = 3.\ 3$$
$$2 \times 1 = 2.\ 2$$
$$3 \times 0 + 1 + 4 = 15.\ 15 = \frac{5}{4}$$
$$3 \times 4 = 12.\ 12$$
$$3 \times 0 + 1 + 4 + 9 = 42.\ 42 = \frac{7}{6}$$
$$4 \times 9 = 36.\ 36$$
$$3 \times 0 + 1 + 4 + 9 + 16 = 90.\ 90 = \frac{9}{8}$$
$$5 \times 16 = 80.\ 80$$
$$3 \times 0 + 1 + 4 + 9 + 16 + 25 = 165.165 = \frac{11}{10}$$
$$6 \times 25 = 125.125$$

As for example, if the terms be two, the triple of the terms will be to the greateſt term drawn into the number of terms as 3 to 2; if there be three terms as 5 to 4; if four, as 7 to 6; if five, as 9 to 8, and ſo continually: So that the antecedents of theſe proportions always mutually exceed one another by the number 2; and ſo every antecedent its conſequent by 1. Whence it is evident that by how much the greater the number of terms is, by ſo much the more the proportion tends to equality. So 100 to 99 is leſs diſtant from the proportion of equality than 10 to 9. From hence, ſuppoſing the number of terms infinite (or infinitely great,) the triple of quantities proceeding thus in a duplicate proportion (or as the ſquares of the numbers, 0, 1, 2, 3, 4, &c. will be equal to as many quantities equal to the greateſt term.

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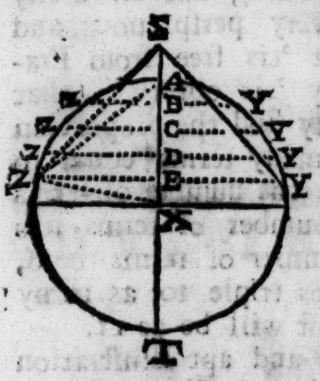
The ſame, as to the ſubſtance of it, is laid down by *Archimedes* in his *Book of Spirals*, as the Foundation of many Argumentations, in that, and other Books, and is well demonſtrated by our Learned Country-man Dr. *Wallis*: However, I thought fit to illuſtrate the matter by this method, as being not unworthy our Conſideration, and very perſpicuous and intelligible in this, that 'tis free from Fractions. And by the way 'tis obſerv'd, that from hence we may eaſily find the proportion of a ſeries triple to as many terms equal to the greateſt, *viz.* as twice the number of terms leſs one, to twice the number of terms leſs two. So that if the number of terms be 6, the proportion of a ſeries triple to as many terms equal to the greateſt will be as 11.

It will be a very eaſy and apt Illuſtration of this Rule, if we infer hence, *That a Cone is ſubtriple of a Cylinder, having an equal baſe and altitude.* For let us ſuppoſe the altitude AE of the cone ZY to be divided into equal and indefinitely many parts, by as many parallel right lines ZY, and the lines ZY will be as the numbers 1, 2, 3, 4, *&c.* and the ſquares or circles conſtituted upon the diameters ZY, as 1, 4, 9, 16, *&c.* whence all thoſe circles, or the whole cone AZY (made up of the ſame) will be ſubtriple of as many circles equal to the greateſt, conſtituted on the greateſt diameter ZEY, that is, ſubtriple of a cylinder whoſe baſe is AEY, and altitude AE.

There occur two other moſt apt examples of this Rule, *viz.* by inferring, *That the complement of a Semiparabola is ſubtriple of a parallelogram having the ſame baſe and height*; as alſo,

alſo, *That the ſpace comprehended by the Spiral and Radius is ſubtriple of the circle in which the Spiral is generated:* But of theſe in another place. Wherefore to go on with what we began, theſe two Rules being ſuppoſed; let us conceive ZAY to be a ſegment of a ſphere, X its center, AT its diameter, and ZAYT a great circle paſſing thro' the vertex, and the part AE of the Axe to be divided into an indefinitely many equal parts; and let us imagine parallel lines to be drawn thro' the points of Diviſion, generating circles in the ſphere, whoſe Radii let be BZ, CZ, DZ, and diameters ZY. I ſuppoſe the ſegment of a ſphere to conſiſt of all theſe parallel circles, whoſe number is as great as that of the points, or equal indefinitely many ſmall parts in the Axe AE, according to the known *Method of Indiviſibles.*



But now for brevity's ſake, let the diameter AT be called d, and the radius of the ſphere r (if need be,) and the Axe AE, by which the number of terms is expreſs'd, call n, and one of the equal parts a; which being ſettled, 'tis evident, (*by the Elements*) that $\overline{BZ}^2 = AB \times BT = a \times d - a = ad - a^2$, and in like manner $\overline{CZ}^2 = AC \times CT = 2a \times d - 2a = 2ad - 4a^2$, and by the ſame reaſoning $\overline{DZ}^2 = AD \times DT = 3ad - 9a^2$, and $\overline{EZ}^2 = AE \times ET = 4ad - 16a^2$, &c. that is, that the ſquares of the radii of the circles ZY are

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to one another as the rectangles ad, $2ad$, $3ad$, $4ad$, &c. (which proceed in an Arithmetical Progreſſion from 0) leſs by the ſquare a^2, $4a^2$, $9a^2$, $16a^2$, &c. which go on as the ſquares of the numbers, 1, 2, 3, 4, &c.) But by our firſt Rule, all the Rectangles 0, ad. $2ad$, $3ad$, $4ad$, &c. are equal to half as many terms equal to the greateſt AE × AT or nd, that is, $= \frac{nd \times n}{2}$.

Moreover, by our ſecond Rule, all the ſquares 0, a^2, $4a^2$, $9a^2$, $16a^2$, &c. taken together, are equal to a third part of as many terms equal to the greateſt $\overline{AE}^2$ or n^2, that is, $= \frac{n^2 \times n}{3}$.

Wherefore all the ſquares deſcribed upon the radii BZ, CZ, DZ, EZ, conjunctly, are equal to the difference $\frac{ndn}{2} \frac{nnn}{3}$, (or the terms being reduc'd to the ſame denomination,) $\frac{3\,ndn - 2\,nnn}{6}$ and their quadruple, that is, all the ſquares deſcribed upon the diameter ZY, are equal to $\frac{12\,ndn - 8\,n^3}{6}$ or $\frac{6\,ndn - 4\,n^3}{3}$.

Whence a ſegment of a ſphere is equal to a Cylinder, the diameter of whoſe baſe is the ſide of a ſquare equal to $6\,nd - 4\,n^2$, and altitude is $\frac{1}{3}n$; or to a cone having the ſame baſe, but the altitude n, or which is all one, having a baſe whoſe radius is $\sqrt{\frac{6\,nd - 4\,n^2}{4}}$ or $\sqrt{\frac{3}{2}\,nd - n^2}$, and altitude n as before. Which Cone we may change into a Cone upon the ſame baſe ZY with the ſegment ZAY, by ſaying, as ZE^2 (*i. e.* $dn - n^2$) to $\frac{3}{2}\,nd - n^2$ or (both

(both terms being divided by n) as $d - n$ to $\frac{3}{2}d - n$, ſo reciprocally n to the altitude of the Cone ſought: Or in the figure by making, *as TE to TE + XA, ſo is EA to ES. For ES will be the altitude of the Cone ZSY equal to the ſegment of the ſphere ZAY.* Which is a noted Theorem of *Archimedes*, demonſtrated by him with ſo much labour and prolixity.

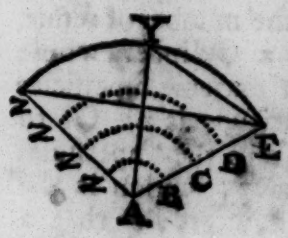
Hence, if the given ſegment be a *Hemiſphere*, and ſo $n = \frac{1}{2}d$ or r, then d or $2r$ will be the altitude of a *Cone*, which having a baſe equal to the baſe of the *Hemiſphere*, (or to the greateſt circle in the ſphere,) will be equal to the *Hemiſphere*. And a *Cone* whoſe baſe is double of the greateſt circle, and the altitude $2r$, or the *Cylinder* whoſe baſe is $\frac{1}{4}$ of the greateſt circle, and altitude $2r$, will be equal to the whole *Sphere*. Whence the whole *Sphere* is $\frac{2}{3}$ of a *Cylinder* the diameter of whoſe baſe is $2r$, and the altitude alſo $2r$. And this is the chief Theorem of *Archimedes*, viz. *That a ſphere is ſubſeſquialter or* $\frac{2}{3}$ *of that Cylinder, whoſe Altitude and Diameter of the baſe is equal to the Diameter of the Sphere.*

Furthermore, not to paſs over any thing in our Author which ſeems to be to our purpoſe:

If to the ſum firſt found, repreſenting a ſegment, *viz.* $\frac{6ndn - 4nnn}{3}$, we add $\frac{2d^2n - 6dn^2 + 4n^3}{3}$ $(= \frac{4dn - 4n^2}{3} \times \frac{d - 2n}{2} = \frac{4}{3}\overline{ZE^2} \times XE)$ repreſenting the Cone ZXY, the aggregate $\frac{2}{3}ddn$ will repreſent the Sector of the ſphere ZXYA, which for that reaſon will be equal to a Cylinder, the diameter of whoſe baſe $\sqrt{dn}$, and

and the altitude $\frac{4}{3}d$, or to a Cone, the diameter of whose base is $\sqrt{dn}$, and the altitude $2d$, or also in a Cone, the Radius of whose base is $\sqrt{dn}$, and the altitude $\frac{1}{2}d = r$ (it being reciprocally as $4dn : dn :: 2d : \frac{1}{2}d$,) that is, to a Cone, the Radius of whose base is the Line AZ, drawn from the vertex to the circumference of the base of the segment, (for $\overline{AZ}^2 = TA \times AE = dn$,) and the altitude r. And this is the next famous Theorem of *Archimedes*, concerning the solidity of the sector of the Sphere, *viz. That the sector of a sphere is equal to a Cone, whose base is a circle described by a Radius equal to a line drawn from the vertex to the circumference of the base of the segment, and whose altitude is equal to the Radius of the sphere.*

And thus I think I have completed that which belongs to the solidity of a sphere, and its parts with sufficient brevity and perspicuity. From hence we shall deduce the Resolution of the other Problem, which I proposed concerning the surface of the segment of a sphere; and then of the whole sphere. To obtain this, as we supposed, before a *Circle* to consist of concentric Peripheries, and the *Sector of a Circle*, of concentric Arcs, (in the number of which, the greatest, and the least or a point is reckon'd: So now we suppose spheres to consist of concentric spherical superficies, and the *Sectors of Spheres*, of like concentric superficies; as for example, the sector of the sphere ZAE, of the superficies BZ, CZ, DZ, EZ, *&c.*) which supposition indeed seems so easy and natural, that in my judgment 'tis sufficient



cient only to propoſe it; neither is a further explication wanting to gain an aſſent to it.

2. We ſuppoſe theſe ſpherical ſuperficies to be in a duplicate Ratio of the Radius of the ſpheres: This is the common affection of all like ſuperficies, and it ſeems to agree very well with the ſuperficies of ſpheres, becauſe they appear to be moſt uniform and ſimilar. But this Suppoſition might eaſily be evinc'd and eſtabliſh'd by the ſame ſort of arguing, as ſpheres are proved to be in triplicate proportion to their Diameters or Radii; Or might have been join'd as a *Corollary* to *Prop.* 17. and 18. *Elem.* 12. where the ſuperficies of like Polygones are ſuppos'd to be inſcribed in ſpheres, having as well the ſuperficies in a duplicate, as the ſolidity in a triplicate Ratio of the Diameters of the Spheres. Theſe things being premis'd, let us ſuppoſe AE a Radius or the ſide of the Sector of a Sphere EAZ, to be divided into equal and indefinitely many ſmall parts, and the ſector AEZ to conſiſt of theſe ſpherical ſuperficies BZ, CZ, DZ, EZ, it will be evident that all thoſe ſuperficies in the Progreſſion are as the ſquares of the Radii, that is, as $\overline{AB}^2$, $\overline{AC}^2$, $\overline{AD}^2$, $\overline{AE}^2$, *&c.* or as the ſquares of the numbers 1, 2, 3, 4, *&c.* whence by our ſecond Rule, the ſum of all theſe ſuperficies, that is, the ſector AEZ, will be $\frac{1}{3}$ of as many ſuperficies equal to the greateſt FZ, that is, $\frac{1}{3}$ of the greateſt EZ, drawn into r the number of terms. Whence a ſector is equal to a Cylinder whoſe baſe is $\frac{1}{3}$ of the greateſt or extreme ſuperficies of the ſector, and whoſe altitude is r: Or to a Cone whoſe baſe is equal to the ſuperficies of the ſector, and its altitude r, which is the laſt of *Lib.* 1.) but we juſt now prov'd that a

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ſector is equal to a Cone whoſe altitude is r, and baſe a circle deſcrib'd by the Radius YE, drawn from the vertex of the ſegment EYZ to the circumference of the baſe. Wherefore, a Cone whoſe altitude is, and baſe equal to the ſuperficies of the ſector, is equal to a Cone of the ſame altitude, whoſe baſe is a circle deſcrib'd by the Radius YE.

And ſo the ſuperficies of the ſector EYZ is equal to a circle deſcrib'd by the Radius YE, Which, certainly is the principal Theorem of all thoſe that occur in the Books of *Archimedes*, nor is there found a more excellent one in all Geometry; *viz. That the ſuperficies of any ſegment of a ſphere is equal to a circle whoſe Radius is a right line drawn from the vertex of the ſegment to the circumference of the baſes:* And hence, *That the ſuperficies of an Hemiſphere is double of the baſe, or equal to two great circles of the ſphere.*

For in this Caſe $\overline{YE}^2 = \overline{AZ}^2 + \overline{AY}^2 = 2\overline{AE}^2$ and conſequently a circle deſcribed by the Radius YE is equal to two circles deſcrib'd by the Radius AE. Whence alſo, *the ſuperficies of the whole ſphere is quadruple a circle having the ſame Radius with the ſphere*, that is, *quadruple the greateſt circle in the ſphere; and equal to a circle whoſe Radius is the diameter of the ſphere.* From hence it follows, *That the ſuperficies of a ſphere is equal to the ſuperficies of a Cylinder of the ſame heighth and breadth*; for the ſuperficies of that Cylinder is quadruple to the baſe, as we ſhall ſhew hereafter. And theſe are the moſt noted Theorems of *Archimedes*. Nay, from hence all thoſe things follow, which he

has written concerning the ſuperficies of ſpheres, and their ſegments. So that from theſe few and eaſy Suppoſitions, I have demonſtrated whatever ſeem to be of any Note in the Books *Of the Sphere and Cylinder.*

I will only add, that after by the method of *Archimedes*, (for I think ſcarce any other can be invented, beſides ours, for finding the ſolidity) the ſuperficies of ſegments are found equal to the circle deſcribed by the Radii YE; hence it will plainly follow, that the ſuperficies of ſpheres, and thence of like ſectors are in a duplicate ratio of the Radii of the ſpheres; and conſequently from the ſuperficies thus found, the contents of ſegments, and of whole ſpheres may be mutually deduced, and that very clearly and expeditiouſly after this manner. Becauſe in the ſector EAZ (fig. Pag. 358.) the ſuperficies BZ, CZ, DZ, EZ, proceed as the ſquares deſcrib'd upon AB, AC, AD, AE, that is, as 1, 4, 9, 16, *&c.* the whole ſector will be equal to $\frac{1}{3}$ of as many ſuperficies equal to the greateſt EZ, or EZ × *r*, that is, to a Cylinder whoſe baſe is $\frac{1}{3}$ EZ, and altitude *r*, or to a Cone whoſe baſe is $\frac{1}{3}$Z, and altitude *r*. But EZ is ſuppoſed equal to a circle whoſe Radius is YE, wherefore the ſector EAZ is equal to a Cone whoſe baſe is a circle deſcribed by the Radius YZ and altitude *r*: Which is *Archimedes*'s univerſal Theorem for the contents of *Circles*. Whence if from this the Cone ZAE ſtanding on the baſe of the ſegment EYZ, and having the vertex at the center of the ſphere A, be ſubducted, (you'll have that ſegment EYZ.) But when the ſector EYZ is a Hemiſphere, there will be no ſuch Cone to be ſubducted; and for that reaſon a Cylinder whoſe baſe is $\frac{2}{3}$ EZ, and altitude *r*, or

the

the Cone whoſe baſe is 2EZ, and altitude likewiſe *r* will be equal to the whole ſphere. But the ſuperficies of the Hemiſphere EZ, is proved to be equal to two of the greateſt circles in the ſphere, whence the whole ſphere is given. This is *Archimedes*'s firſt and principal Theorem, for the content of a ſphere; whence 'tis eaſily deduced, that a ſphere is $\frac{2}{3}$ of a circumſcrib'd Cylinder, that is, of a Cylinder whoſe altitude and diameter of its baſe is equal to the diameter of the ſphere.

The Doctrine of our other [*Archimedes*] ſeems to make againſt, and ſubvert the new and celebrated *Method of Indiviſibles*, and is preſs'd to that end by *Tacquet*, for inſtance, (*Prop.* 2. *lib.* 2. *Cylindr.*) For the uſual proceſs of that method ſeems to exhibit the dimenſion of the ſuperficies of a Cone, (as alſo of a ſphere, and of other Curves) different enough from what our Author and others have demonſtrated: As for example, let us ſuppoſe ABCD a right Cone, whoſe Axe is AX, and baſe BCD, and plane $\beta\chi\delta$ drawn, at pleaſure, parallel to the baſe BCD. And ſince, as *Diam.* BD : *Periph.* BCD :: *Diam.* $\beta\delta$: *Periph.* $\beta\chi\delta$, and ſo every where it will be (according to the *Method of Indiviſibles*, and by 12. 5.) as *Diam.* BD, to *Periph.* BCD, ſo is the triangle ABD, conſiſting of thoſe parallel Diameters, to the *Conic Superficies* ABCD, conſiſting of thoſe Peripheries, *i. e. Diam.* BD : *Periph.* BCD :: $\frac{AX \times BD}{2}$: $\frac{AX \times \textit{Periph.}\ BCD}{2}$. Whence

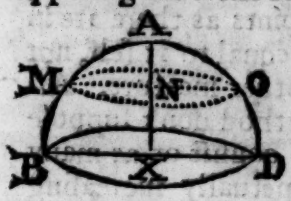
$\frac{AX \times Periph.\ BCD}{2}$ will be equal to the ſuperficies of the Cone; which is falſe and contrary to what was demonſtrated juſt now. For we demonſtrated that the ſuperficies of the Cone was $\frac{AB \times Periph.\ BCD}{2}$.

In anſwering this Objection, we ſay, that the *Method of Indiviſibles*, in the ſpeculation of Perimeters, and of Curve Surfaces, proceeds otherwiſe than in the ſpeculation of plane Surfaces and ſolid Contents. It does indeed ſuppoſe that the Area of plane Figures conſiſt, as it were of parallel right lines, and the contents of ſolids of parallel Planes, and that their number may be expreſs'd by the altitude of the Figures: But it by no means ſuppoſes, that the Perimeters of plane figures conſiſt of points, or the ſuperficies of ſolids, of lines, the number of which may be expreſs'd by the altitude of the figure. As for example, altho' the triangle ABD (in the laſt figure) conſiſts of lines parallel to BD, the number of which is expreſſed by the number of points in the perpendicular AX, that is, by the length of the perpendicular: Yet it would be abſurd to ſuppoſe that the line AB conſiſts of points, whoſe number may be expreſs'd by the number of points in a leſs line AX. For altho' the right line $\beta\xi$ drawn thro' each infinitely ſmall part of AX, divide AB into as many infinitely ſmall parts, yet thoſe parts are not of the ſame Denomination or Quality with the parts of AX, but ſomewhat greater than them; ſo that if the parts of AX be look'd upon as points, the parts of AB are not to be called points, but greater than points; and on the contrary, if the parts of AB be called

points,

points, the parts of AX are to be look'd upon as leſs than points, if it be lawful to ſpeak ſo. For the points which are treated of in the *Method of Indiviſibles* are not abſolutely points, but indefinitely ſmall parts, which uſurp the names of points, becauſe of the Affinity. Since therefore points don't admit of greater and leſs, the name of points is not at the ſame time to be attributed to the parts of different magnitudes; conſequently tho' the number of the greater parts of AB may be expreſs'd by the number of the leſſer parts of AX, yet the number of points in AB can no ways be expreſſed by the number of points in AX, (that is, by the number of parts in AX, equal to the number of parts in AB, which are called points.) The line AB has as many points as there are in it ſelf alone, or another line equal to it ſelf, nor can it be determin'd by any other meaſure. After the ſame manner, this method don't ſuppoſe the conic Surface ABCD to conſiſt of as many parallel circumferences perpetually increaſing from the vertex A, or decreaſing from the baſe BD, as there are points in the Axe AX; but rather of as many thus increaſing or decreaſing as there are points in the ſide AB. For in the Revolution of the line AB about the Axis AX, (whereby the ſuperficies of the Cone is generated) every point in the line AB produces a circumference, and conſequently more circumferences are produced than the points contained in the Axis AX. Therefore if you would extend the *Method of Indiviſibles* to the ſuperficies of ſolids, and ſuppoſe thoſe ſuperficies to conſiſt of parallel lines, you ought not to compute this by the parallel Areas conſtituting the ſolid, that is, not to number thoſe Areas by the altitude of the ſolid, but by other lines agreeable to the condition of each figure. Which lines, in figures that are not irregular, may eaſily be determin'd: For inſtance,

ſtance, in the equilateral Pyramid ABCD, whoſe Axe is AX, ſuppoſing that the lateral ſurface of the Pyramid conſiſts of Perimeters of triangles, parallel to the baſe BCD, theſe can neither be computed by the altitude AX, nor by the ſide AB, (for by the former, the thing requir'd, would be wanting of the true Dimenſion, and by the latter 'twould exceed it,) but by the line AE drawn from the vertex A perpendicular to the ſide BC of the baſe; the reaſon of which is, that every plane ſide of a Pyramid as ABC, conſiſts of parallel right lines computed by the altitude AE. After the ſame manner, ſuppoſing that the ſuperficies of the Hemiſphere BAD, conſiſt of peripheries of circles parallel to the baſe BCD, the number of them is not to be computed by the Axis AX, but by the Quadrantal Arc AB, becauſe that every point of the Arc AD in revolving produces a circumference. And ſo any ſuperficies, whether plane or curv'd, which is conceived to conſiſt of equidiſtant right or curv'd lines, is to be computed by a line cutting thoſe equidiſtant lines perpendicularly. For ſince thoſe equidiſtant lines, in this *Method of Indiviſibles*, are not conſider'd abſolutely as lines having an infinitely ſmall breadth, which is the ſame with the breadth or thickneſs of the point deſcribing thoſe equidiſtant lines in their Circumvolution, and ſince the ſame equidiſtant lines divide the line cutting them perpendicularly into parts meaſuring its breadth, thoſe parts are to be look'd upon as ſuch ſort of points, and conſequently the number of equidiſtant lines, or the ſum of thoſe breadths is to be computed by the number of points in the line cutting



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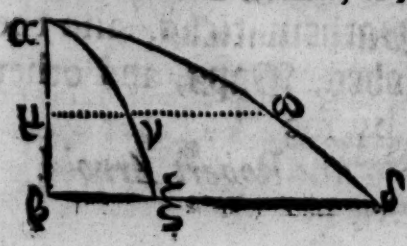
cutting them perpendicularly, that is, by the length of that line, and not by a line of any other length, for that will consist of more or less points.

Hence therefore in the speculation of the superficies of solids, the *Method of Indivisibles* is not unuseful, but rather very commodious, provided it be rightly understood, and applied according to the Rule prescrib'd. For by the help of it even these superficies may be found, if so be we have some convenient *Data* presuppos'd, on which the reasoning may be founded. For instance, we might by the help of it, investigate the superficies of a Cone, by reasoning after this manner.

If the superficies of the cone ABC (fig. pag. 362.) be divided into innumerable Peripheries of circles $\beta\chi\delta$ parallel to the base BCD, the breadth of those Peripheries taken together, make up the side AB cutting them perpendicularly, and consequently there will be as many Peripheries as there are points in the line AB, that is, their number may be express'd by the number of points in AB, or by its length. Wherefore, if you draw perpendiculars equal to the Peripheries to every point of AB, a superficies will be made out of those perpendiculars equal to the superficies of the Cone. But that superficies will be a triangle whose heighth is AB, and base equal to the greatest Periphery BDC, and so the superficies of the Cone will be $= \frac{1}{2}$ AB $\times$ *Periph.* BDC, which conclusion agrees with the things laid down and demonstrated by *Archimedes*.

After the same manner, if you take any right line $\alpha\beta$ equal to the quadrantal Arc AB of the Hemisphere (*in pag.* 364.) and to each of its points μ let the per-



perpendiculars $\mu\nu$ be erected equal to the Radii MN of parallel circles MOM passing thro' the corresponding points M of that quadrantal Arc, the greatest of which $\beta\xi$ let be equal to the Radius BX of the base of the Hemisphere: The figure $\alpha\beta\xi$ will contain the Radii of all the circles of whose Peripheries the superficies of the Hemisphere consists. And if the perpendicular $\omega\omega$, $\beta\delta$ be erected equal to the Peripheries MOM, BDB, there will be made the figure $\alpha\beta\delta$ equal to the superficies of the Hemisphere. The dimension of which figure if you can by any means find (as in this case you are to find the Area of the figure $\alpha\beta\xi$) thence you will easily deduce the content of the segment of the sphere, agreeing to what you may gather by any lawful reasoning. Which Observation, I think will not be unuseful in *Geometry*.

FINIS.

AT the *Hand* and *Pen* in *Barbican* are Taught, **Writing**, **Common** and **Merchants Accounts**, after that well approved Method of Mr. *Normansel*, **Algebra**, Geometry, **Measuring**, Surveying, Gauging, **Navigation** and Dialing, with other parts of the **Mathematicks**, also the Use of the **Globes**, **Maps**, and other Instruments, by me

Robert Arnold.

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